Construction of energetic particle fluxes due to Alfvénic fluctuations from parallel mode structures

M. V. Falessi[1], L. Chen[2,3,1], Z.Qiu[2,1], F. Zonca[1,2]

[1] Center for Nonlinear Plasma Science and ENEA C. R. Frascati, Frascati, Italy

[2] Inst. for Fusion Theory and Simulation and Department of Physics Zhejiang University, Hangzhou 310027, China

[3] Dept. Physics and Astronomy, University of California, Irvine, CA 92697-4575, USA

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Introduction: Outline

- Introduction;
- Fluid theory;
- Kinetic theory;
- Summary & Conclusions.

Introduction: MET motivations and objectives

- Predicting the dynamics of a burning plasma on long time scales, comparable with the energy confinement time or longer, is essential in order to understand next generation fusion experiments, e.g. ITER;
- the crucial role of energetic particles as mediators of cross scale couplings, see Zonca et al. 2015; Chen and Zonca 2016, must be properly described;

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- usual transport analysis must be extended to the phase space because of the underlying kinetic nature of wave-particle interactions and fluctuation excitation;
- proper structures describing such transport processes consistently with usual transport theories are the phase space zonal structures (PSZS);
- MET project provides a viable route to computing energetic particle phase space transport in burning plasmas on long (transport) time scales, based on nonlinear gyrokinetic theory;

Introduction: MET hierarchical approach

- the zeroth level of simplification consist in the gyrokinetics description of plasma dynamics;
- the first level of simplification consist in assuming $|\omega| \gg \tau_{NL}^{-1} \sim \gamma_L$;
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- in recent works, i.e. Zonca et al. 2020, the intermediate level of this hierarchy has been introduced, i.e. the Dyson Schrödinger Model (DSM);
- the nonlinear envelope equations describing the SAW fluctuation spectrum and the PSZS transport equations have been introduced;

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- in recent works, i.e. Zonca et al. 2020, the intermediate level of this hierarchy has been introduced, i.e. the Dyson Schrödinger Model (DSM);
- the nonlinear envelope equations describing the SAW fluctuation spectrum and the PSZS transport equations have been introduced;
- beyond the applicability of the wave kinetic equation and of the radially local description, e.g. flux-tube or quasilinear approaches;
- this approach is general and can be applied to 3D magnetic equilibria (see the talk by A. Zocco in this meeting) and space plasmas;

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- this is the essential ingredient for the calculation of energetic particle fluxes (see the talk by F. Zonca in this meeting);
- first step toward the solution of the DSM and the investigation of structure formation in strongly magnetized toroidal plasmas;
- all the terms, even the nonlinear ones, can be calculated by means of linear physics;

Notation and fundamental equations

- We assume an axisymmetric equilibrium magnetic field $\boldsymbol{B}_0$ in flux coordinates:

$$\boldsymbol{B}_0 = F(\psi)\boldsymbol{\nabla}\varphi + \nabla\phi \times \nabla\psi$$

- The perpendicular plasma displacement:

$$\delta\boldsymbol{\xi}_\perp = \frac{c}{B_0}\boldsymbol{b} \times \boldsymbol{\nabla}\Phi_s \ ,$$

- can be written in terms of the perturbed stream function:

$$\Phi_s(r,\theta,\zeta) = \sum_m \exp(in\zeta - im\theta)\Phi_{sm}(r)$$

Notation and fundamental equations

- Following Lu, Zonca, and Cardinali 2012, we decompose Φ_s:

$$\begin{aligned} \Phi_s(r,\theta,\zeta) &= 2\pi \sum_{\ell\in\mathbb{Z}} e^{in\zeta - inq(\theta-2\pi\ell)} \hat{\Phi}_s(r,\theta-2\pi\ell) \\ &= \sum_{m\in\mathbb{Z}} e^{in\zeta - im\theta} \int e^{i(m-nq)\vartheta} \hat{\Phi}_s(r,\vartheta) d\vartheta \quad . \end{aligned}$$

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- ϑ represents the extended poloidal angle coordinate following equilibrium magnetic field lines;
- as a simple illustrative application of the present formalism we recall the continuous spectrum calculation in the fluid limit . . .

Fluid theory: SAW/ISW continuous spectrum

- …Following Chen and Zonca 2016 and the recent Falessi et al. 2019; Falessi et al. 2020, continuous spectra are obtained from vorticity and pressure balance equations when $|\vartheta| \to \infty$;

$$\left(\partial_\vartheta^2 - \frac{\partial_\vartheta^2 |\nabla r|}{|\nabla r|} + \frac{\omega^2 J^2 B_0^2}{v_A^2}\right) y_1 = (2\Gamma\overline{\beta})^{1/2} \kappa_g \frac{J^2 B_0 \overline{B}_0}{q R_0} \frac{s\vartheta}{|s\vartheta|} y_2$$

$$\left(1 + \frac{c_s^2}{\omega^2} \frac{1}{J^2 B_0^2} \partial_\vartheta^2\right) y_2 = (2\Gamma\overline{\beta})^{1/2} \kappa_g \frac{\overline{B}_0}{B_0} q R_0 \frac{s\vartheta}{|s\vartheta|} y_1$$

where:

$$y_1 \equiv \frac{\hat{\phi}_s}{\left(\overline{\beta} q^2\right)^{1/2}} \frac{c k_\vartheta}{\bar{B}_0 R_0}, \quad y_2 \equiv i \frac{\delta \hat{P}_{comp}}{(2\Gamma)^{1/2} P_0},$$

- $\hat{\phi}_s(r,\vartheta) \equiv |s\vartheta| |\boldsymbol{\nabla} r| \hat{\Phi}_s(r,\vartheta)$, $s = rq'/q$ is the magnetic shear, $k_\vartheta = -nq/r$, $\overline{\beta} = 8\pi\Gamma P_0/\overline{B}_0^2$, $\hat{\Phi}_s$ and $\delta\hat{P}_{comp}(r,\vartheta)$ are the representation of the perturbed stream function and the compressional component of the pressure perturbation;

Fluid theory: SAW/ISW continuous spectrum

- linear system of second order ODEs with periodic coefficients;
- Floquet theory demonstrate that it must have solutions in the form:

$$\mathbf{x}_i = e^{i\nu_i\vartheta}\mathbf{P}_i(\vartheta)$$

- where $\mathbf{P}_i$ is a 2π-periodic function $i = 1, 2, 3, 4$ and the ν_i are the characteristic Floquet exponents.

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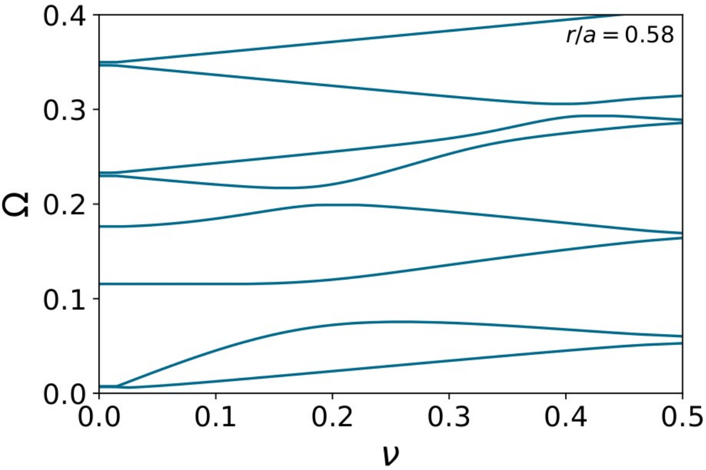
- where $\mathbf{P}_i$ is a 2π-periodic function $i = 1, 2, 3, 4$ and the ν_i are the characteristic Floquet exponents.
- Solving the system for a given r thus calculating ν_i for each ω we obtain:

$$\nu_i = \nu_i(\omega, r)$$

- this relation involves only local quantities and describes wave packets propagation along magnetic field lines;

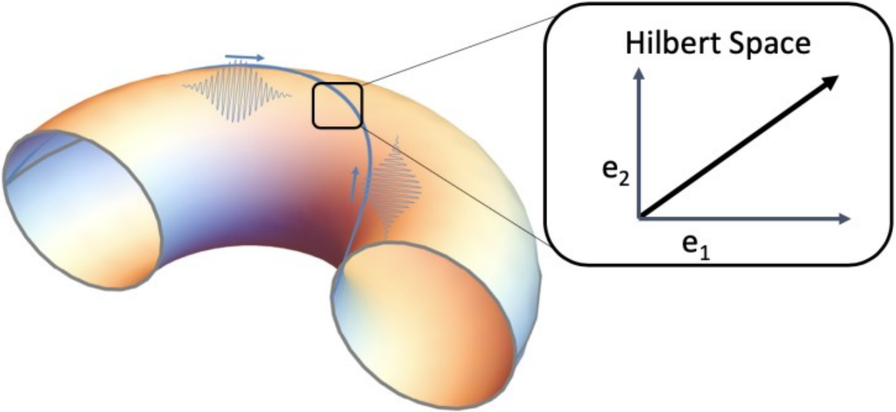
Fluid theory: Local dispersion relation



- local dispersion curves $\nu(\Omega)$ calculated by FALCON for a Divertor Tokamak Test (DTT) reference scenario;

Fluid theory: Local dispersion relation

- Any fluctuation with short scale (large- ϑ) radial structure satisfy the same equations;
- therefore it is a linear superposition of Floquet solutions;



Fluid theory: Polarization

- In Falessi et al. 2020; Falessi et al. 2019 we define the concept of polarization of a Floquet solution:

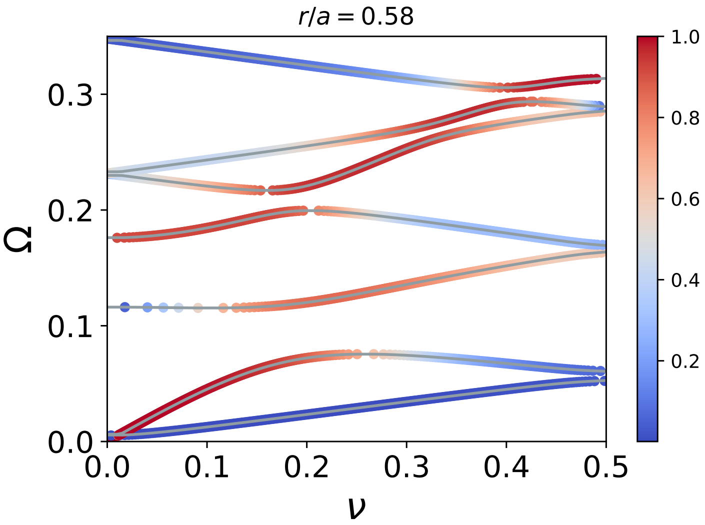
$$
\begin{aligned}
y_1^{(i)}(\vartheta;\nu_i,r) &= e_1^{(i)}(\nu_i,r)\hat{y}_1^{(i)}(\vartheta;\nu_i,r)\ , \\
y_2^{(i)}(\vartheta;\nu_i,r) &= e_2^{(i)}(\nu_i,r)\hat{y}_2^{(i)}(\vartheta;\nu_i,r)\ .
\end{aligned}
$$

with the following normalizations:

$$
\int d\vartheta |\hat{y}_1^{(i)}|^2 = \int d\vartheta |\hat{y}_2^{(i)}|^2 = 1\ , \quad |e_1^{(i)}(\nu_i,r)|^2 + |e_2^{(i)}(\nu_i,r)|^2 = 1\ .
$$

- mode polarization is crucial in order to assess the actual coupling with the continuous spectrum and resonant absorption;
- polarization is an integral over ϑ of the solutions;

Fluid theory: Polarization



- polarization results calculated by `FALCON` for DTT;
- envelope equation coefficients can be calculated similarly ...

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- following Zonca and Chen 2014a; Zonca and Chen 2014b, we introduce the formalism of wave equations in slowly evolving weakly non uniform media:

$$\begin{pmatrix} \frac{\hat{\phi}_s}{(\beta q^2)^{1/2}} \frac{ck_\vartheta}{B_0 R_0} \\ i\frac{\delta \hat{P}_{\text{comp}}}{(2\Gamma)^{1/2} P_0} \end{pmatrix} \equiv A(r,t) e^{iS(r)} \begin{pmatrix} e_1(r,t) y_1(\vartheta; r, t) \\ e_2(r,t) y_2(\vartheta; r,t) \end{pmatrix}$$

where we have introduced the radial envelope $A_n(r,t)$ and the eikonal $S(r,t)$;

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  where we have introduced the radial envelope $A_n(r,t)$ and the eikonal $S(r,t)$;
- pressure and vorticity equations reads:

$$\mathcal{L}_A e_1 y_1 - \left(2\Gamma \beta q^2\right)^{1/2} \frac{g(\vartheta, \theta_k)}{\hat{\kappa}_\perp} e_2 y_2 = 0$$
$$\mathcal{L}_S e_2 y_2 - \left(2\Gamma \beta q^2\right)^{1/2} \frac{g(\vartheta, \theta_k)}{\hat{\kappa}_\perp} e_1 y_1 = 0$$

where $\theta_k = -i/(nq')\partial_r A$.

Fluid theory: Non radially singular response

- the WKB eigenvalue problem can then be written as:

$$\boldsymbol{D}\left(r, t, k_r, \omega\right) \cdot \boldsymbol{A}(r, t) e^{iS(r,t)} = 0$$

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- matrix elements are calculated integrating the solutions of this system along ϑ:

$$D_{11}\left(r,t,k_r,\omega\right)=\int_{-\infty}^{\infty}y_1^*\mathcal{L}_A y_1 d\vartheta$$
$$D_{12}\left(r,t,k_r,\omega\right)=D_{21}^*\left(r,t,k_r,\omega\right)=\int_{-\infty}^{\infty}\left(2\Gamma\beta q^2\right)^{1/2}\frac{g(\vartheta,\theta_k)}{\hat{\kappa}_\perp}y_1^* y_2 d\vartheta$$
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- parallel mode structure must have been solved integrating the wave equation with proper boundary conditions;

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- parallel mode structure must have been solved integrating the wave equation with proper boundary conditions;
- this defines a dispersion relation $\omega=\Omega(k_r,r,t)$;
- the same approach can be applied to study the kinetic problem . . .

Kinetic theory: Governing equations for EM potentials

$$\sum \left\langle \frac{e^2}{m} \frac{\partial \bar{F}_0}{\partial \mathcal{E}} \right\rangle_v \delta\phi + \nabla \cdot \sum \left\langle \frac{e^2}{m} \frac{2\mu}{\Omega^2} \frac{\partial \bar{F}_0}{\partial \mu} \left(\frac{J_0^2 - 1}{\lambda^2} \right) \right\rangle_v \nabla_\perp \delta\phi + \sum \left\langle e J_0(\lambda) \delta g \right\rangle_v = 0$$

$$B_0 \left(\nabla_\parallel + \frac{\delta B_\perp}{B_0} \cdot \nabla \right) \left(\frac{\delta J_\parallel}{B_0} \right) - \nabla \cdot \sum \left\langle \frac{e^2}{m} \frac{2\mu}{\Omega^2} \left(B_0 \frac{\partial \bar{F}_0}{\partial \mathcal{E}} + \frac{\partial \bar{F}_0}{\partial \mu} \right) \left(\frac{J_0^2 - 1}{\lambda^2} \right) \right\rangle_v \nabla_\perp \frac{\partial}{\partial t} \delta\phi + \ldots = 0$$

$$\nabla_\perp \left(B_0 \delta B_\parallel + 4\pi \delta P_\perp \right) \simeq 0$$

- the governing equations for $\delta A_\parallel$, $\delta\phi$ are the usual (nonlinear) quasineutrality and vorticity equations reviewed in Chen and Zonca 2016;

Kinetic theory: Governing equations for EM potentials

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- the governing equations for $\delta A_\parallel$, $\delta\phi$ are the usual (nonlinear) quasineutrality and vorticity equations reviewed in Chen and Zonca 2016;
- $\bar{F}_0$ is an arbitrary (renormalized) anisotropic distribution function that includes PSZS contribution;
- for describing $n \neq 0$ fluctuations instead of $\delta A_\parallel$ we introduce $\delta\psi$ where $\nabla_\parallel \delta\psi \equiv -(1/c)\partial_t \delta A_\parallel$;

Kinetic theory: Governing equations for EM potentials

- following the previous methodology, we introduce:

$$\begin{pmatrix} e\delta\hat{\psi}_n(r,\vartheta;t)/T_{0i} \\ e\delta\hat{\phi}_{\parallel n}(r,\vartheta;t)/T_{0i} \end{pmatrix} \equiv A_n(r,t)e^{iS_n(r,t)} \begin{pmatrix} e_1(r,t)y_1(r,\vartheta;t) \\ e_2(r,t)y_2(r,\vartheta;t) \end{pmatrix}$$

- from the EM potential equations we readily obtain:

$$\hat{\boldsymbol{e}}_n^+ \cdot \boldsymbol{D}\left(r,t,k_{nr},\omega_n\right) \cdot \boldsymbol{A}_n(r,t)e^{iS_n(r,t)} = \hat{\boldsymbol{e}}_n^+ \cdot \boldsymbol{F}(r,t)\ ,$$

- $\boldsymbol{F}(r,t)$ denotes all nonlinear interactions and external forcing ...

Kinetic theory: The Dyson Schrödinger Transport Model

- we now introduce the main assumption of the model:

$$|\gamma_{Ln}| \sim \tau_{NLn}^{-1} \ll |\omega_n|$$

- that is that the non-linear characteristic time scale is very long compared to ω^{-1};

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$$|\gamma_{Ln}| \sim \tau_{NLn}^{-1} \ll |\omega_n|$$

- that is that the non-linear characteristic time scale is very long compared to ω^{-1};
- consistently, y_1, y_2 can be approximated by the corresponding linear parallel mode structures;
- D can be calculated at any order of this asymptotic expansion, e.g. the lowest order local dispersion relation reads:

$$D_{Rn}^0 (r, t, k_{nr}, \omega_n) \equiv \hat{\boldsymbol{e}}_n^+ \cdot \boldsymbol{D}_R^0 \cdot \hat{\boldsymbol{e}}_n = 0$$

- while the radial envelope equation reads ...

Kinetic theory: The Dyson Schrödinger Transport Model

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial D^0_{Rn}}{\partial \omega_n}A_n^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D^0_{Rn}}{\partial k_{nr}}A_n^2\right) + 2D^1_{An}A_n^2 - 2iD^1_{Rn}A_n^2 \\
&+iA_n\left(\frac{\partial^2 D^0_{Rn}}{\partial k_{nr}^2} + 2\frac{\partial \hat{\boldsymbol{e}}_n^+}{\partial k_{nr}}\cdot \boldsymbol{D}^0_{Rn}\cdot \frac{\partial \hat{\boldsymbol{e}}_n}{\partial k_{nr}}\right)\frac{\partial^2 A_n}{\partial r^2} = -2ie^{-iS_n}A_n\hat{\boldsymbol{e}}_n^+\cdot \boldsymbol{F} \\
&-\left(\hat{\boldsymbol{e}}_n^+\cdot\frac{d}{dt}\hat{\boldsymbol{e}}_n - \frac{d}{dt}\hat{\boldsymbol{e}}_n^+\cdot\hat{\boldsymbol{e}}_n\right)\frac{\partial D^0_{Rn}}{\partial \omega_n}A_n^2 + \left(\frac{\partial \hat{\boldsymbol{e}}_n^+}{\partial \omega_n}\cdot \boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{\boldsymbol{e}}_n}{\partial t} - \frac{\partial \hat{\boldsymbol{e}}_n^+}{\partial t}\cdot \boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{\boldsymbol{e}}_n}{\partial \omega_n}\right)A_n^2 \\
&-\left(\frac{\partial \hat{\boldsymbol{e}}_n^+}{\partial k_{nr}}\cdot \boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{\boldsymbol{e}}_n}{\partial r} - \frac{\partial \hat{\boldsymbol{e}}_n^+}{\partial r}\cdot \boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{\boldsymbol{e}}_n}{\partial k_{nr}}\right)A_n^2 \,.
\end{aligned}
$$

- beyond the standard wave kinetic equation;

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&-\left(\frac{\partial \hat{\boldsymbol{e}}_n^+}{\partial k_{nr}}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{\boldsymbol{e}}_n}{\partial r} - \frac{\partial \hat{\boldsymbol{e}}_n^+}{\partial r}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{\boldsymbol{e}}_n}{\partial k_{nr}}\right)A_n^2 \;.
\end{aligned}
$$

- beyond the standard wave kinetic equation;
- wave packets can be focused/defocused and back scattered by both nonlinearities as well as by radial nonuniformities;
- additional terms of fundamental importance in the description of EP induced avalanches, see Zonca et al. 2005, but also for the interaction of zonal fields and drift wave turbulence, see Guo, Chen, and Zonca 2009;

Kinetic theory: Computation of DSM coefficients

- $\boldsymbol{F}$ takes the form of a weighted average along the linear parallel mode structure of the nonlinear term describing nonlocal interactions in the mode number space;

$$\hat{\boldsymbol{e}}^{+} \cdot \boldsymbol{F} = e_1^*(r) \int_{-\infty}^{\infty} d\vartheta y_1^* \left[Vort\right]^{NL} + e_2^*(r) \int_{-\infty}^{\infty} d\vartheta y_2^* \left[QN\right]^{NL}$$

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- continuum calculation, e.g. such as the `FALCON/FALCON-K` results, constitute the boundary conditions in the ballooning space, i.e. $\vartheta \to \pm\infty$, to determine $y_1, y_2, \boldsymbol{e}_1, \boldsymbol{e}_2$;
- `DAEPS` code already solve for these structures (see the next talk by Y. Li);
- `FALCON/FALCON-K` has been used for a proof of principle calculation of parallel mode structures and a comparison with `DAEPS` is foreseen;
- all the coefficients appearing in the NLS equation are calculated accordingly;

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- non perturbative W-P interactions can modify lowest order plasma dispersion properties;
- phase space fluxes appearing in the PSZS equation depend on turn on the em potentials (see the talk by F. Zonca) that can be calculated solving the envelope equation;

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- the role of PSZS in the calculation of parallel mode structures is crucial and they must evolve consistently with the radial envelope equation;
- non perturbative W-P interactions can modify lowest order plasma dispersion properties;
- phase space fluxes appearing in the PSZS equation depend on turn on the em potentials (see the talk by F. Zonca) that can be calculated solving the envelope equation;
- the DSM constitute a viable reduced transport model to characterize energetic particle transport on long time scales in realistic tokamak plasmas;

Summary & Conclusions

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- we have shown how this theory describes also perturbations that are non radially singular;
- the same formalism can be applied to the kinetic equations;
- we have described how to calculate the coefficients appearing in the envelope equation of the DSM model using only linear parallel mode structures;
- linear codes, e.g. `DAEPS`, `LIGKA`, `FALCON/FALCON-K` can provide the required inputs;
- moving towards the first DSM simulation in realistic equilibria!

Thank you for your attention.