Transport in the beam-plasma system: self-consistency issue and extension of the QL model

Updates on WP2

Mid-term MET Workshop, 23rd March 2020

Outline

- Transport hierarchical scheme: Vlasov-Poisson system.
- Relevance of the self-consistency.
- Extension of the QL model.
- Deliverable of WP2.

The Vlasov-Poisson system

- Interaction: **energetic electron beam** and cold background plasma.
- Kinetic description in Fourier space - **Beam distribution function** $f(t,v,x)$ normalized to $N_B = n_B L$ (1D periodic slab $0 \leqslant x \leqslant L$):

$$n_B L = \int f(t,v,x) dv\, dx \qquad f_B(t,v) \equiv \int f(t,v,x) dx$$

- **Vlasov-Poisson coupled system** - Fourier components of electric field $E_k(t)$ and of the distribution function $f_k(t,v)$. It reads

$$\partial_t f_k = -ikvf_k + \frac{e}{m}\sum_q E_{k-q}\partial_v f_q \qquad \partial_t E_k = -i\omega_p E_k + \frac{2\pi e\omega_p}{k}\int dv\, f_k$$

- $\boldsymbol{f_0}$: only component having non-zero initial condition (spatial homogeneity), **main transport properties of the system**:

$$\partial_t f_0 - \frac{e}{m}\sum_k \left[E_k^* \partial_v f_k + E_k \partial_v f_k^*\right] = 0$$

- Validation of the hierarchical transport module -
  Direct integration using **explicit expression of fluxes:**
  $\Rightarrow$ **sample $\boldsymbol{E_k}$** and $\boldsymbol{f_k}$ from simulations.

  Results exactly match the fully self-consistent N-body system:

$$\partial_\tau \bar{f}_B(\tau, u) = \partial_u \Big[4\pi \sum_\ell \Big[\ell \bar{\phi}^b_\ell \, \bar{f}^a_\ell - \ell \bar{\phi}^a_\ell \, \bar{f}^b_\ell \Big] \Big]$$

  - Analog of this scheme has been verified for EP/AE transport flux expression with HMGC $\rightarrow$ Falessi talk.

- Normalization: $\bar{x} = x(2\pi/L)$, $\tau = t\omega_p$, $u = v(2\pi/L)/\omega_p$, $\ell = k(2\pi/L)^{-1}$.
  Dimensionless distribution $\bar{f}(\tau, u, \bar{x})$ (**histograms** of the phase-space):

$$f_0(t, v) = \frac{n_B(2\pi/L)}{\omega_p M} \bar{f}_B(\tau, u) \qquad M = \int \bar{F}_B(u) du$$

Diagonal reduction

- Assumption: $\boldsymbol{f_k}$ receives mainly contribution by the **correspondent harmonics** (neglect mode-mode interaction):

$$\partial_t f_k = -ikv\, f_k + \frac{e}{m} E_k \partial_v f_0$$

Substituting $f_k(t,v) = \frac{e}{m}\int_0^t dt' E_k(t') e^{ikv(t'-t)} \partial_v f_0(t',v)$ into the f_0 Vlasov equation (cc-notation: $k > 0$) :

$$\partial_t f_0(t,v) - \frac{e^2}{m^2} \sum_k \left[E_k^* \, \partial_v \Big(\int_0^t dt'\, E_k(t') e^{ikv(t-t')} \partial_v f_0(t',v) \Big) + c.c. \right] = 0$$

- Equation base of **reduced transport model hierarchy** for 1D beam-plasma system.

Self-consistent Dyson-like system

- Without loss of generality, the electric field can be set as

$$E_k(t) \propto \exp\left[-i\int_0^t dt'\,\omega_k(t')\right] \qquad \Rightarrow \qquad E_k(t') = E_k(t)\,\exp\left[-i\int_t^{t'} dy\,\omega_k(y)\right]$$

- Substituting into reduced Vlasov eq (+Poisson): **Dyson-like system**

$$\boldsymbol{\partial_t f_0(t,v)} = \frac{e^2}{m^2}\sum_{k>0}|E_k(t)|^2\partial_v\Big(\int_0^t dt'\,\exp\Big[ikv(t'-t)-i\int_t^{t'}dy\,\omega_k(y)\Big]\partial_v f_0(t',v)+c.c.\Big)$$

$$\boldsymbol{\partial_t |E_k|^2} = \frac{2\pi e^2\omega_p}{mk}|E_k|^2\Big(\int_{-\infty}^{\infty}dv\int_0^t dt'\,\exp\Big[ikv(t'-t)-i\int_t^{t'}dy\,\omega_k(y)\Big]\partial_v f_0(t',v)+c.c.\Big)$$

Fully self-consistent scheme (1D analogous of WP2.1-M1).

External fields

- Approximation Level 1: direct integration for **given spectrum** $E_k(t)$ **from simulation only**.
- The diagonal reduced Vlasov equation

$$\partial_t f_0(t,v) - \frac{e^2}{m^2}\sum_k \left[E_k^* \,\partial_v\Big(\int_0^t dt'\, E_k(t') e^{ikv(t-t')}\partial_v f_0(t',v)\Big) + c.c.\right] = 0$$

can be manipulated to obtain the (normalized) **transport equation**:

$$\partial_\tau \bar{f}_B(\tau,u) = \partial_u \sum_\ell \ell^2(\bar{\phi}_\ell \bar{G}_\ell^* + \bar{\phi}_\ell^* \bar{G}_\ell)$$
$$\partial_\tau \bar{G}_\ell(\tau,u) = -i\ell u \bar{G}_\ell + \bar{\phi}_\ell \partial_u \bar{f}_0$$

- $\bar{G}_\ell$: spectral components of the distribution function. Tabulated $\bar{\phi}_\ell(\tau)$ from sims: RK ($4th$) evolves the system in time. Initial conditions: $\bar{f}_B(0,u) = \bar{F}_B(u)$ (Gaussian-bump), $\bar{G}_\ell(0,u) = 0$.

External fields: self-consistency issue

[NC, F. Finelli, G. Montani, *Europhys. Lett.* **127**, 25002 (2019)]

- **Single constant mode** - Under the diagonal assumption, and using the diagram summation applied to the Dyson formulation:

$$\hat{f}_0(\omega,\xi) = \frac{iF_0(\xi)}{\omega} - \alpha^2\partial_\xi \frac{1}{\omega^2-\alpha^2\xi^2}\partial_\xi \hat{f}_0(\omega,\xi) \quad \alpha^2 = \sqrt{2}ek|E_k^{sat}|/m$$

  where $\xi = (k/\alpha)(v-\omega_k^R/k)$.

- Defining $\Psi(\omega,\xi) = \alpha^2(\omega^2-\alpha^2\xi^2)^{-1}\partial_\xi\hat{f}_0(\omega,\xi)$:

$$\partial_\xi^2\Psi(\omega,\xi) + \left(\omega^2\alpha^{-2}-\xi^2\right)\Psi(\omega,\xi) = \frac{i}{\omega}\partial_\xi F_0(\xi)$$

  closely resembling the equation for parabolic cylinder functions (PCFs):

$$\partial_\xi^2\psi_n(\xi) + (2n+1-\xi^2)\psi_n(\xi) = 0 \qquad \psi_n(\xi) = \frac{e^{-\xi^2/2}}{\sqrt{2^n n!\sqrt{\pi}}}H_n(\xi)$$

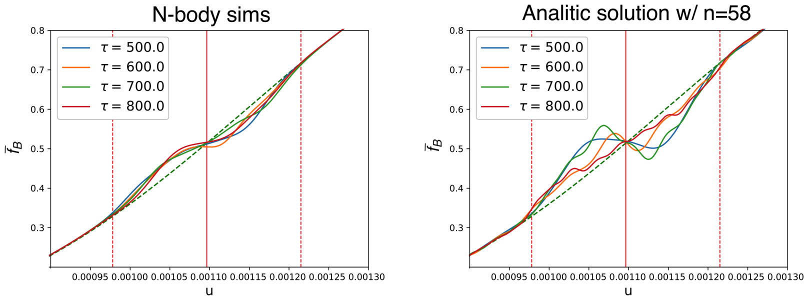
  where PCFs are an orthonormal basis and $H_n(\xi)$ are the Hermite pols.

- Analytic solution of the diagonal reduced Vlasov equation:

$$f_B(t,v) = F_B(v) + \sum_{n\geqslant 0} \frac{\beta_n}{2n+1} \partial_v \psi_n(v) [1 - \cos(\alpha\sqrt{2n+1}\, t)]$$

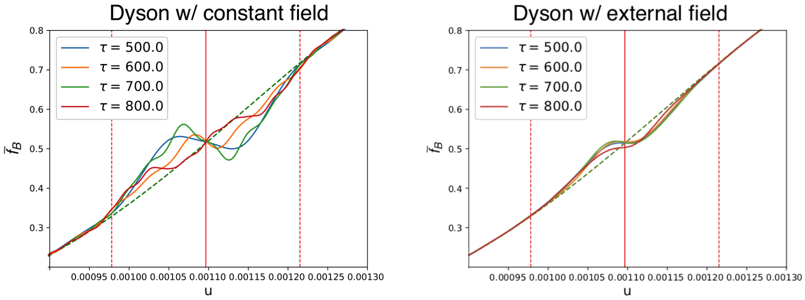
written in terms of PCFs ψ_n and with

$$\beta_n = \int dv\, \psi_n(v) \partial_v F_B(v)$$



- Analytic sol. well describe resonance position and non-linear spread, but generates corrugation and gradient inversion.

- Integration of reduced Vlasov equation: transport eq. $\partial_t f_B = \partial_v \Gamma$.
  In the case of **external constant mode**: same results.



As soon as we consider the **self-consistent mode** extracted from dynamics, the morphology well resembles the numerical simulation.

$\Rightarrow$ Shortcoming of the analytical sol is **NOT** related to the truncated expansion, but the constant mode assumption $\Rightarrow$ **self-consist. importance.**

Quasi-linear model

- Approximation level 2:
  - f_B (and ω_k) **not fast varying** in time;
  - **broad** (and **dense**) **spectrum**: $\sum_k (...) \to \int_{k_{min}}^{k_{\max}} dk\, \mathcal{N}(k)(...)$ with the spectral density $\mathcal{N} = m/\Delta k$ (Δk is the spectral width).

- Self-consistent Dyson eqs (diagonal reduced VP system) rewrite:

$$\partial_t f_0(t,v) = \partial_v(\mathcal{D}_{QL}(t,v)\partial_v f_0) \qquad \mathcal{D}_{QL} = \frac{e^2\pi\mathcal{N}}{m^2 v}|E(t,v)|^2$$

$$\partial_t |E(t,v)|^2 = 2\gamma_{QL}(t,v)|E|^2 \qquad \gamma_{QL} = \frac{2\pi^2 e^2 v^2}{m\omega_p}\partial_v f_0$$

$\to$ **Velocity space**: continuous spectrum $|E(t,k)| \to |E(t,\omega_p/v)|$ from the discrete one, specified by resonance conditions (set $\omega_k \simeq \omega_p$).

- Dimensionless variables - $\mathcal{I}(\tau, u) = |\phi|^2$ (spectrum), $\bar{\mathcal{N}} = m/\Delta\ell$:

$$\partial_\tau \bar{f}_B(\tau, u) = \partial_u \left[\frac{\pi \bar{\mathcal{N}}}{u^3} \partial_u f_B \, \mathcal{I}_0 \; \exp[H] \right]$$
$$\partial_\tau H(\tau, u) = \frac{\pi \eta}{M} u^2 \, \partial_u \bar{f}_B$$

  - Parallel formulation wrt spectral PDE: precise representation of the discrete simulated mode spectrum (smoothing for $\partial_u \mathcal{I}$)
  - Initial conditions: $\bar{f}_B(0, u) = \bar{F}_B(u)$, $\bar{H}(0, u) = 0$.

- **Spectral evolution**:

$$\mathcal{I}(\tau, u) = \mathcal{I}_0 \; \exp[H] \qquad \qquad \mathcal{I}_0 = \mathcal{I}(0, u)$$

Extension of QL model

[G. Montani, F. Cianfrani, NC, *Plasma Phys. Contr. Fusion* **61**, 075018 (2019)]

- VP reduced system:

$$\boldsymbol{\partial_t f_0(t,v)} = \frac{e^2}{m^2}\sum_{k>0}|E_k(t)|^2\partial_v\Big(\int_0^t dt' \exp\Big[ikv(t'-t) - i\int_t^{t'} dy\,\omega_k(y)\Big]\partial_v f_0(t',v) + c.c.\Big)$$

$$\boldsymbol{\partial_t |E_k|^2} = \frac{2\pi e^2\omega_p}{mk}|E_k|^2\Big(\int_{-\infty}^{\infty} dv \int_0^t dt' \exp\Big[ikv(t'-t) - i\int_t^{t'} dy\,\omega_k(y)\Big]\partial_v f_0(t',v) + c.c.\Big)$$

- Integral in dt' can be rewritten as

$$\int_0^t dt' \frac{\partial_v f_0(t',v)}{i(kv-\omega_k(t'))}\ \partial_{t'}\Big(\exp\Big[ikv(t'-t) - i\int_t^{t'} \omega_k(y)dy\Big]\Big)$$

- As QL model: assuming a **dense Langmuir spectrum** $kv = \omega_k$ $[\forall v]$.

- **Taylor expansion** $\partial_v f_0(t', v) = \partial_v f_0(t, v) + (t' - t)\,\partial_t \partial_v f_0(t, v)$. We also assume $\omega_k = \omega_p + \delta\omega_k \simeq \omega_p$ (the exponent is almost vanishing):

$$\int_0^t dt' \frac{\partial_v f_0(t,v)}{i(kv - \omega_k(t'))} \partial_{t'} \Big(\exp\Big[i(t'-t)\delta\omega_k - i\int_t^{t'} \delta\omega_k(y) dy \Big] \Big) + \\ - \int_0^t dt' \frac{\partial_t \partial_v f_0(t,v)}{i(kv - \omega_k(t'))} \exp\Big[i(t'-t)\delta\omega_k - i\int_t^{t'} \delta\omega_k(y) dy \Big]$$

- As QL model, assuming ω_k **not fast varying in time**, the $t = 0$ contribution results to be exponentially small [O'Neil 71]:

$$\frac{\partial_v f_0(t,v)}{i(kv - \omega_k(t))} + \frac{\partial_t \partial_v f_0(t,v)}{(kv - \omega_k(t))^2}$$

- Evolution of the **distribution function** (before broad spectrum assumption, $\sum_k (...) \to \int dk\, \mathcal{N}(k)(...)$ with $\mathcal{N} = const.$):

$$\boldsymbol{\partial_t f_0 = \partial_v (\mathcal{D}_{QL}(t,v) \partial_v f_0 + \mathcal{D}_1(t,v) \partial_t \partial_v f_0)} \qquad \mathcal{D}_{QL}(t,v) \equiv \frac{e^2}{m^2} \sum_{k>0} |E_k|^2 \Big[\frac{1}{i(kv - \omega_k)} + c.c. \Big]$$

$$\mathcal{D}_1(t,v) \equiv \frac{e^2}{m^2} \sum_{k>0} \mid E_k(t) \mid^2 \Big[\frac{1}{(kv - \omega_k)^2} + \frac{1}{(kv - \omega_k^*)^2} \Big]$$

- Using the same approach, **Poisson equation** can be restated as:

$$\partial_t |E_k(t)|^2 = 2\Gamma_k |E_k|^2$$

$$\boldsymbol{\Gamma_k(t) = \gamma_k^{QL} + \delta\gamma_k}$$

$$\delta\gamma_k(t) = \frac{\pi e^2 \omega_p}{mk} \int_{-\infty}^{+\infty} dv \left[\frac{1}{(kv - \omega_k)^2} + \frac{1}{(kv - \omega_k^*)^2} \right] \partial_t \partial_v f_0$$

  the QL growth rate γ_k^{QL} is obtained for $\mathrm{Im}(\omega_k) \ll \omega_p$.
- The non-resonant contribution to the growth rate $\delta\gamma_k$ can be evaluated as (details in the PPCF paper:)

$$\delta\gamma_k = -\frac{4(\sqrt{2}-1)\pi e^2 \omega_p}{mk^2 \Gamma_k} \partial_t \partial_v f_0 |_{v=\omega_p/k}$$

- Equation for Γ_k:

$$\Gamma_k = \gamma_k^{QL} - \frac{4(\sqrt{2}-1)\pi e^2 \omega_p}{mk^2 \Gamma_k} \partial_t \partial_v f_0 |_{v=\omega_p/k}$$

- Using dimensionless variable (barred growth rates are in ω_p units):

$$\partial_\tau \bar{f}_B(\tau,u) = \partial_u(\bar{\mathcal{D}}_{QL}\partial_u \bar{f}_B) + \partial_u(\bar{\mathcal{D}}_1 \partial_\tau \partial_u \bar{f}_B)$$
$$\mathcal{D}_{QL}(\tau,u) = \pi\bar{\mathcal{N}}\mathcal{I}/u^3$$
$$\mathcal{D}_1(\tau,u) = 6\bar{\mathcal{N}}\, u\, \bar{\Gamma}\, \partial_u^2(\mathcal{I}/u^2)$$

$$\partial_t \mathcal{I}(\tau,u) = 2\bar{\Gamma}\mathcal{I}$$
$$\bar{\Gamma}(\tau,u) = \bar{\gamma}_{QL}/2\left(1+\sqrt{1-4\bar{\delta}^2/\bar{\gamma}_{QL}^2}\right)$$
$$\bar{\gamma}_{QL}(\tau,u) = \frac{\pi\eta u^2}{2M}\partial_u \bar{f}_B$$
$$\bar{\delta}^2(\tau,u) = \frac{(\sqrt{2}-1)\eta u^2}{M}\partial_\tau\partial_u \bar{f}_B$$

Model valid for **short times** $(t'-t) \ll 1$ (time scale before flattening: avoid the **divergence** in the instantaneous growth rate $\bar{\Gamma}(\tau,u)$).

- Expansion of $\bar{\Gamma}(\tau, u)$ for small perturbations, *i.e.*, for $4\bar{\delta}^2/\bar{\gamma}_{QL}^2 \ll 1$:

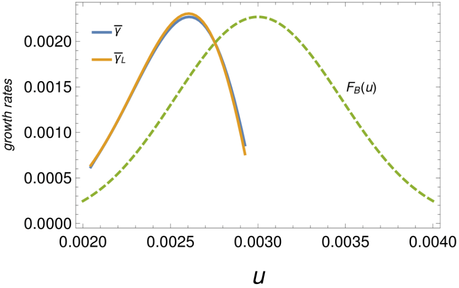
$$\bar{\Gamma}(\tau, u) = \bar{\gamma}_{QL} - \bar{\delta}^2/\bar{\gamma}_{QL}$$

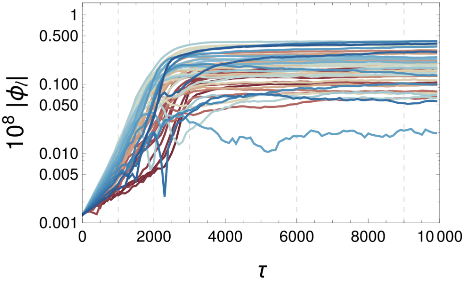
- Using $\mathcal{I} = \mathcal{I}(0, u) \exp\Big[\int_0^\tau 2\bar{\Gamma}(\tau', u) d\tau'\Big]$ and taking the primitive of the exact differentiation ($\bar{\delta}^2/\bar{\gamma}_{QL} \sim \partial_\tau ln[\partial_u \bar{f}_B]$):

$$\frac{\mathcal{I}}{\mathcal{I}_{QL}} = \Big(\frac{|\partial_u \bar{F}_B|}{|\partial_u \bar{f}_B|}\Big)^{4(\sqrt{2}-1)/\pi}$$

During the flattening, $\partial_u \bar{f}_B$ clearly decreases with respect to the Gaussian initial condition: **enhancing of the predicted spectrum** with respect to the QL one.

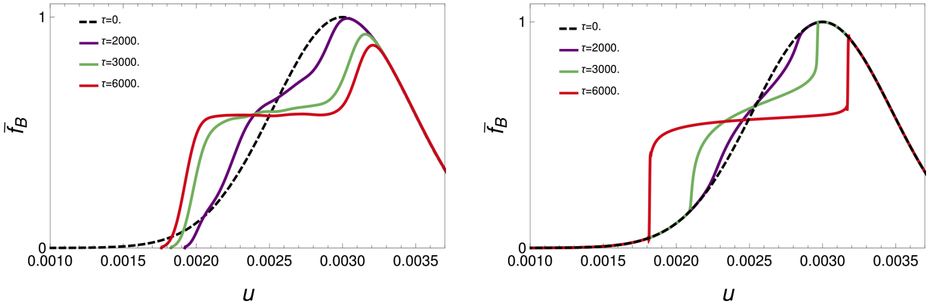
- Reference case: **broad spectrum**, 45 modes, $\tau_{ac}/\tau_B \simeq 0.02$.
  - QL approximation for $\mathcal{K}_D = \tau_{ac}/\tau_B \ll 1$: particles feel all spectrum before trapping time $\rightarrow$ (random walk); diffusion-like process.





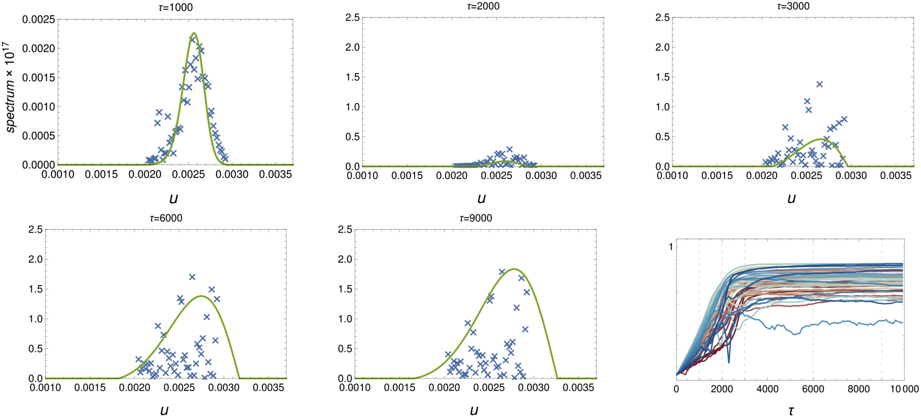
- Numerically evolved modes (Hamiltonian system) have growth rate values well predicted by the Landau expression.

- **Distribution function**: N-body vs. QL model



- Time delay in the QL evolution: slower formation of the plateau (hypothesis of slow evolution of f_0).
- Flattening discrepancy: QL plateau is larger. No fixed set of modes.

Spectral evolution: *N*-body vs. QL model

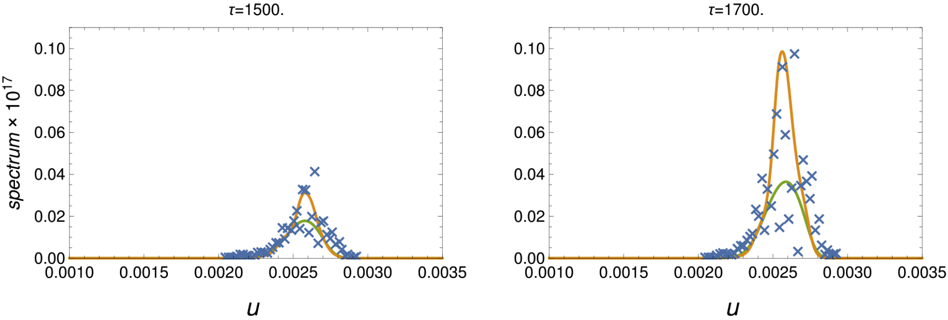


- Field intensity is very small: recover the linear growth of the intensity (standard inverse Landau damping rate).
- QL model is not predictive in the temporal mesoscales: diffusive-like behavior not present in the real simulations.
- Mixed diffusion/convection.
- Diffusive contribution induces a too early termination of the linear regime (spectrum remains at a lower level).
- Flattening width discrepancy: QL spectrum is larger (fixed number of simulated modes).

- **Spectral evolution**: Extended vs. standard QL model

  Early temporal mesoscales: extraction of distribution function from the N-body dynamics ($\partial_u \bar{f}_B$) to evaluate:

$$\frac{\mathcal{I}}{\mathcal{I}_{QL}} = \left(\frac{|\partial_u \bar{F}_B|}{|\partial_u \bar{f}_B|}\right)^{4(\sqrt{2}-1)/\pi}$$



- Amended spectrum (orange) is significantly close to the numerical one (original QL spectrum in green).
- First order correction to the f_0 slow varying assumption enhance the growth rate: cure the spectral evolution

Deliverable of WP2

- Transport analysis (WP2.2)
  - understand the relaxation dynamics of beam-plasma systems beyond the oft-used quasi-linear approximation
  - investigate the conditions motivating the possible breakdown of the diffusive style paradigm and a pathway to non-Gaussian, non-diffusive transport, abandoning the familiar quasi-linear picture of diffusion on the asymptotic time scales
  - develop simplified EP transport models in support of whole device modeling

$\rightarrow$ Proposed an extension of QL model: cross-coupling terms among spectral components are still neglected (possible revised picture (?)).

$\rightarrow$ Importance of convective transport (breakdown of diffusive paradigm) also addressed in the comparison wrt HAGIS-LIGKA sims.

$\rightarrow$ 1D transport module (almost) fully addressed. Help in defining peculiar physics for the general scheme (*i.e.*, spectral self-consistency).

- Numerical implementation of PSZS transport equations (WP2.1)
  - development of transport module (1D space; 2D GK vel. space) for phase space transport analysis
  - Best solution to be discussed by MET Team (modification of existing module? new one?): novel aspect is explicit implementation of fluxes in the EP phase space and of corresponding evolution of the fluctuation spectrum with multi-level approach (WP1 and WP3; AWECS).
  - Implementation of general geometry in AWECS

$\rightarrow$ First approximation level started with HMGC code (Falessi talk).

$\rightarrow$ Numerical implementation of the flux transport equation.

$\rightarrow$ FALCON to calculate parallel mode structure (Falessi talk).