# Energetic particle nonlinear equilibria and reduced models for transport processes in burning plasmas

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The views and opinions expressed herein do not necessarily reflect those of the European Commission.

# Outline

- EP Transport equations; WP1
- Zonal state; WP1
- Hierarchical approach; WP1 - WP2
- Falcon Code. WP2

# EP transport equation: purpose statement

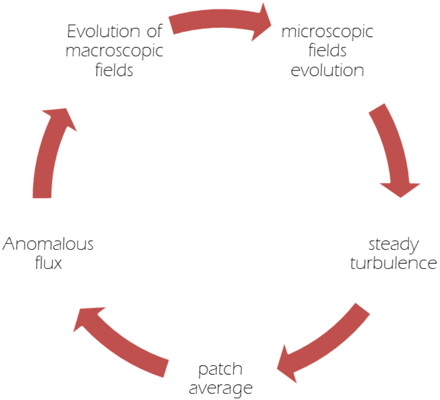
We aim at providing the general expressions describing EP dynamics on long time scales (transport) and at introducing a framework to solve these equations within different levels of reduced dynamics.

By means of this approach it will be possible to:

- describe the physics of next generation fusion experiments where alpha particles will play a key role;
- characterize cross-scale couplings in burning plasmas.

# EP Transport equations: multiple scales analysis

- A multiple scales approach has been applied to extend gyrokinetic simulations to long time scales, see e.g. Trinity;
- macro-scales characterize radial plasma profiles, i.e. $L, \tau_T$, while micro-scales describe the “fast” variation of physical quantities, i.e. $\rho_L, \omega^{-1}$;
- transport equations for radial profiles are derived by means of a systematic separation of scales between macroscopic equilibrium and fluctuations.
- intermediate spatio-temporal scales are suppressed. Is it possible to use the same approach retaining meso-scales produced by the dynamics?;



# EP Transport equations: why meso-scales are important?

## Meso-scales on ITER . . .

- peculiar feature of ITER: interplay of meso-scale structures with micro-scales generated by energetic particles, i.e. $\rho_{LE} \sim (\rho_L L)^{1/2}$;
- unique role of energetic particles, which act as mediators between the micro- and the macro- scales;

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## . . . and DTT (Divertor Tokamak Test facility)

- $T_E/T_i \sim (4\rho*)^{-1} \Rightarrow \rho_{LE} \sim (\rho_L L)^{1/2}$ with dimensionless parameters close to ITER;
- using minority heating by ICRH and/or NBI, cross-scale couplings should be similar;

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- need for transport equations for the mesoscales of the system;
- non Maxwellian distribution functions must be taken into account;
- meso-scales do not emerge only due to EP physics, e.g. ITBs, L-H transitions;
- the derivation, see Falessi 2017; Falessi and Zonca 2019, is based on the theory of Phase space zonal structures, see Chen and Zonca 2016.

# EP Transport equations: ordering assumptions

- In the core the plasma is magnetized, thus:

$$f = F_0 + \delta f \qquad \rho_L \boldsymbol{\nabla} \ln F_0 \sim \rho_L / L \sim \mathcal{O}(\delta)$$

- following Hinton and Hazeltine 1976, Frieman and Chen 1982, we describe slowly varying reference states, e.g. $\omega^{-1}\partial_t \ln p_0 \sim \mathcal{O}(\delta^2)$;
- The reference magnetic field is: $\boldsymbol{B}_0 = F\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi$;
- We introduce the following gyrokinetic ordering for fluctuating quantities:

$$\frac{cE}{Bv_{th}} \sim \frac{|\partial/\partial t|}{|\Omega|} \sim \left|\frac{\delta B}{B}\right| \sim \frac{\boldsymbol{\nabla}_\parallel}{\boldsymbol{\nabla}_\perp} \sim \frac{\boldsymbol{k}_\parallel}{\boldsymbol{k}_\perp} \sim \mathcal{O}(\delta).$$

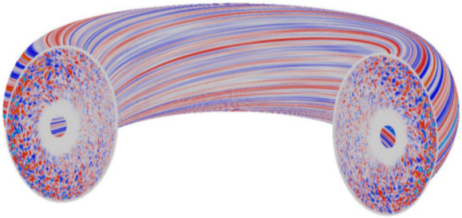

Figure: Courtesy of Y. Xiao et al., PoP 2015.

# EP Transport equations: toroidal symmetric particle response

Considering $\partial_\mu \bar{F}_0 = 0$, and the low$-\beta$ tokamak ordering, we define the zonal structure response (with null toroidal mode number):

$$\delta f_z = e^{-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}} \delta \bar{G}_z + \frac{e}{m} \delta\phi_z \frac{\partial \bar{F}_0}{\partial \mathcal{E}}$$

- $\delta \bar{G}_z$ evolves according to the nonlinear gyrokinetic equation, see Frieman and Chen 1982; Brizard and Hahm 2007:

$$\left(\partial_t + v_\parallel \nabla_\parallel + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}\right) \delta \bar{G}_z = -\frac{e}{m} \frac{\partial \bar{F}_0}{\partial \mathcal{E}} \frac{\partial}{\partial t} \left\langle \delta\psi_{gc} \right\rangle_z - \frac{c}{B_0} \boldsymbol{b} \times \boldsymbol{\nabla} \left\langle \delta\psi_{gc} \right\rangle \cdot \boldsymbol{\nabla} \delta \bar{G} \Big|_z$$

where $\left\langle \delta\psi_{gc} \right\rangle_z = \hat{I}_0 \left(\delta\phi_z - \frac{v_\parallel}{c} \delta A_{\parallel z}\right) + \frac{m}{e} \mu \hat{I}_1 \delta B_{\parallel z}$.

- starting from this expression we will derive an equation describing the transport of particles up to the energy confinement time.

# EP Transport equations: drift/banana center decomposition

- Introducing the (drift/banana center) decomposition $\delta\bar{G}_z = e^{-iQ_z}\delta\bar{g}_z$ and imposing:

$$Q_z = F(\psi)\left[\frac{v_\parallel}{\Omega} - \overline{\left(\frac{v_\parallel}{\Omega}\right)}\right]\frac{k_z}{d\psi/dr}$$

where $k_z \equiv (-i\partial_r)$, $\overline{[\ldots]} \equiv \tau_b^{-1}\oint\frac{d\ell}{v_\parallel}[\ldots]$ is the bounce average

we obtain the following equation for $\delta\bar{g}_z$:

$$\left(\partial_t + v_\parallel\nabla_\parallel\right)\delta\bar{g}_z = e^{iQ_z}\left(-\frac{e}{m}\frac{\partial\bar{F}_0}{\partial\mathcal{E}}\frac{\partial}{\partial t}\left\langle\delta\psi_{gc}\right\rangle_z - \frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\left\langle\delta\psi_{gc}\right\rangle\cdot\boldsymbol{\nabla}\delta\bar{G}\right) .$$

- the requirement for the phase space zonal structure to be long lived imposes that $\nabla_\parallel\delta\bar{g}_z = 0$;
- therefore $\delta\bar{g}_z$ must be toroidally symmetric and characterized by $m = 0$;
- the equation becomes:

$$\partial_t\delta\bar{g}_z = \overline{\left[e^{iQ_z}\left(-\frac{e}{m}\frac{\partial\bar{F}_0}{\partial\mathcal{E}}\frac{\partial}{\partial t}\left\langle\delta\psi_{gc}\right\rangle_z - \frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\left\langle\delta\psi_{gc}\right\rangle\cdot\boldsymbol{\nabla}\delta\bar{G}\right)\right]}.$$

# EP Transport equations: density transport

- Re-writing the low frequency (transport) component of $\delta f_z$ in term of $\delta \bar{g}_z$, taking the time derivative of the surface averaged velocity integral (see Falessi 2017; Falessi and Zonca 2019), we obtain the following:

$$\partial_t \left\langle \langle \delta f_z \rangle_v \right\rangle_\psi = \frac{e}{m} \left\langle \left[ 1 - \overline{\left( e^{-iQ_z} \hat{I}_0 \right)} \, \overline{\left( e^{iQ_z} \hat{I}_0 \right)} \right] \frac{\partial \bar{F}_0}{\partial \mathcal{E}} \partial_t \delta \phi_z \right\rangle_v + \\ - \frac{1}{V'} \frac{\partial}{\partial \psi} \left\langle \left\langle V' \overline{\left( e^{-iQ_z} \hat{I}_0 \right)} \, \overline{\left[ c e^{iQ_z} R^2 \boldsymbol{\nabla} \phi \cdot \boldsymbol{\nabla} \langle \delta \psi_{gc} \rangle \, \delta \bar{G} \right]} \right\rangle_v \right\rangle_\psi$$

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- this equation describes the radial oscillations on any length-scale of the density profile in the absence of collisions and assuming GK ordering;
- mesoscales are spontaneously created by turbulence;
- reference state is never assumed to be Maxwellian.

# EP Transport equations: results (WP 1)

- we have effectively redefined plasma reference state including mesoscales spontaneously produced by the dynamics and derived the related phase space transport equations;
- results known in literature are recovered considering the special case of long wavelength PSZS and Maxwellian reference state;
- theoretical framework able, in principle, to describe the peculiar role of energetic particles on DTT and ITER;
- main results are summarized in a couple of PoP papers: Falessi and Zonca 2019; Falessi and Zonca 2018;
- PSZS and their electromagnetic counterparts allow to define the concept of zonal state . . .

# Zonal state: orbit averaging

- particle motion in the reference magnetic field is characterized by three integrals of motion, i.e. $P_\phi, \mu, \mathcal{E}$;
- the phase space zonal structure equation is connected with the macro- meso-scopic component, i.e. $[\ldots]_S$, unperturbed orbit-averaged distribution function (Falessi et al. 2019b):

$$\frac{\partial}{\partial t}\overline{F_{z0}} + \frac{1}{\tau_b}\left[\frac{\partial}{\partial P_\phi}\overline{\left(\tau_b \delta \dot{P}_\phi \delta F\right)_z} + \frac{\partial}{\partial \mathcal{E}}\overline{\left(\tau_b \delta \dot{\mathcal{E}} \delta F\right)_z}\right]_S = \overline{\left(\sum_b C_b^g\left[F, F_b\right] + \mathcal{S}\right)_{z\,S}}$$

where $\overline{(...)} = \tau_b^{-1} \oint d\theta/\dot{\theta}\,(\ldots) = \oint d\theta/\dot{\theta}\, e^{iQ}(\ldots)(\bar{\psi}, \theta)$, with $\tau_b = \oint d\theta/\dot{\theta}$ and $\psi = \overline{\psi} + \delta\tilde{\psi}(\theta)$;

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- this is equivalent to bounce averaging a quantity shifted with the $e^{iQ_z}$ operator;
- this expression describe transport processes in the phase space due to fluctuations or collisions and sources;
- transport equations for $n, \boldsymbol{P} \ldots$ can be obtained integrating in the velocity space, i.e. $\mathcal{E}, \mu, \alpha$ with proper weights;

# Zonal state: neighboring nonlinear equilibria

- we can decompose the toroidally symmetric distribution function;
- PSZS, i.e. $\overline{F_{z0}}$, from orbit averaging describe macro- & meso-scales;
- micro-scales are accounted by $\delta \bar{F}_z$;

$$F_z = \overline{F_{z0}} + \overline{\delta F_z} + \delta \tilde{F}_z$$

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- micro-scales are accounted by $\delta\bar{F}_z$;
- they describe system transitions between neighboring nonlinear equilibria, see Chen and Zonca 2007; Falessi and Zonca 2019;
- nonlinear equilibria, together with zonal fields, form a zonal state, see Falessi and Zonca 2019.

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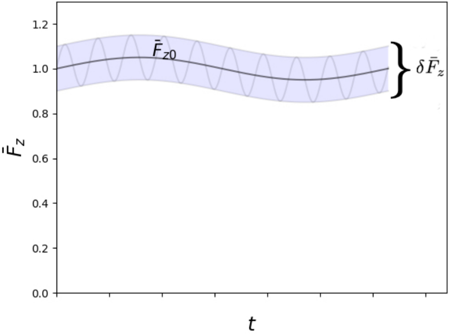

# Zonal state: governing equations

$$\frac{\partial}{\partial t}\overline{F}_{z0} + \frac{1}{\tau_b}\left[\frac{\partial}{\partial P_\phi}\overline{\left(\tau_b \delta \dot{P}_\phi \delta F\right)_z} + \frac{\partial}{\partial \mathcal{E}}\overline{\left(\tau_b \delta \dot{\mathcal{E}} \delta F\right)_z}\right]_S = \overline{C^g_{z0}} + \langle \bar{S} \rangle_S$$

- expanding the collision operator we obtain:

$$\bar{C}^g_{z0} = \overline{C^g_z\left[\overline{F}_{z0}, \overline{F}_{z0}\right]} + \left\langle \overline{C^g_z\left[\overline{F}_{z0}, \delta F_z\right]} + \overline{C^g_z\left[\delta F, \delta F\right]} \right\rangle_S$$

- first term balance $\langle \bar{S} \rangle_S$ while the second, in the presence of a Maxwellian reference state, describes neoclassical transport;
- corrections to neoclassical transport are given by the first and the third terms;

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$$+ \frac{1}{\tau_b}\left\langle \frac{\partial}{\partial P_\phi}\overline{\left(\tau_b \delta \dot{P}_\phi \delta F\right)_z} + \frac{\partial}{\partial \mathcal{E}}\overline{\left(\tau_b \delta \dot{\mathcal{E}} \delta F\right)_z} \right\rangle_F = \overline{C^g}_z - \overline{C^g}_{z_0} + \langle \bar{S} \rangle_F$$

# Zonal state: governing equations

- Computing the zonal state require equations for zonal fields, i.e. the long lived component of toroidal symmetric fields;
- following Chen and Zonca 2016, they are obtained from:

$$\sum\left\langle\frac{e^2}{m}\frac{\partial\bar{F}_{0z}}{\partial\mathcal{E}}\right\rangle_v\delta\phi_z+\nabla\cdot\sum\left\langle\frac{e^2}{m}\frac{2\mu}{\Omega^2}\frac{\partial\bar{F}_{0z}}{\partial\mu}\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v\nabla_\perp\delta\phi_z+\sum\langle eJ_0(\lambda)\delta G_z\rangle_v=0$$

$$\frac{\partial}{\partial t}\delta A_{\parallel z}=\left(\frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\delta A_\parallel\cdot\boldsymbol{\nabla}\delta\psi\right)_z$$

$$\nabla_\perp\delta B_{\parallel z}=\kappa_0\delta B_{\parallel z}+\nabla_\parallel\delta\mathbf{B}_{\perp z}+\nabla\mathbf{b}_0\cdot\delta\mathbf{B}_{\perp z}+\frac{4\pi}{c}\delta\mathbf{J}_{\perp z}$$

where $\delta G_z=\delta F_z-\frac{e}{m}\frac{\partial\bar{F}_0}{\partial\mathcal{E}}\langle\delta\psi_{gcz}\rangle$ and $\boldsymbol{J}_z$ is expressed in terms of $F_z$ using its push-forward representation, see Brizard and Hahm 2007.

# Zonal state: recent developments (WP 1)

- PSZS are a possible definition of nonlinear equilibrium and together with zonal fields and residual orbit averaged particle response they define the zonal state;
- this definition is particularly important in collisionless burning plasmas, where one cannot always describe transport via evolution of macroscopic radial profiles of a reference Maxwellian;
- phase space transport equations in the presence of collisions and sources have been formulated in terms of orbit average;
- the theoretical framework has been presented at the EPS Conference on Plasma physics, AAPPS Conference on Plasma physics, 2nd Trilateral EP Workshop and at the Italian National Conference on the Physics of Matter;

# Zonal state: future developments (WP 1)

- include explicit expressions for orbit averaged sources and collisions;
- derive a closed set of equations (once $n \neq 0$ fluctuations are given) describing the zonal state evolution;
- apply the framework to describe EGAM dynamics;
- main results will be summarized in a NJP paper;
- two contributions will be presented: at the IAEA FEC 2020 and at Varenna;

# Hierarchical approach: transport module

- A transport module will be required to find the numerical solution of phase space transport equations in the presence of collisions and sources. This problem is 1D in space and 2D in velocity space instead of 5D;
- the transport module must be capable of handling different level of reduced dynamics, i.e. different expressions for the fluxes calculated from the output of a kinetic code;

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- the transport module must be capable of handling different level of reduced dynamics, i.e. different expressions for the fluxes calculated from the output of a kinetic code;

- the zeroth level of simplification consist in the gyrokinetics description of plasma dynamics;
- the first level of simplification consist in assuming $|\omega| \gg \tau_{NL}^{-1} \sim \gamma_L$;
- the second and final level of simplification is the Quasilinear model.

# Hierarchical approach: transport module

- e.g. retaining only the effect of non linear diagonal interactions, see Zonca et al. 2015, consistent with $|\omega| \gg \tau_{NL}^{-1} \sim \gamma_L$, we obtain:

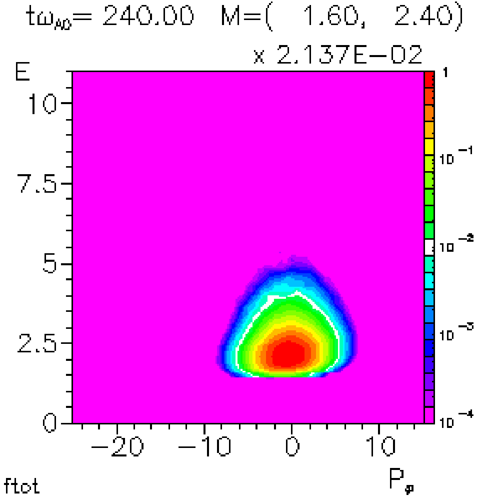
$$\partial_t \overline{F}_{z0} \sim \frac{n^2 c^2}{2} \Im\mathrm{m} \sum_k \frac{\partial}{\partial \psi} \left[ \overline{e^{-iQ} \left\langle \delta\psi_{gc} \right\rangle_k e^{iQ}} \Big|_\psi^* \mathcal{P}_k^{-1} \, \overline{e^{iQ} \left\langle \delta\psi_{gc} \right\rangle_k e^{-iQ}} \Big|_\psi \frac{\partial}{\partial \psi} \overline{F}_{z0} \right]$$

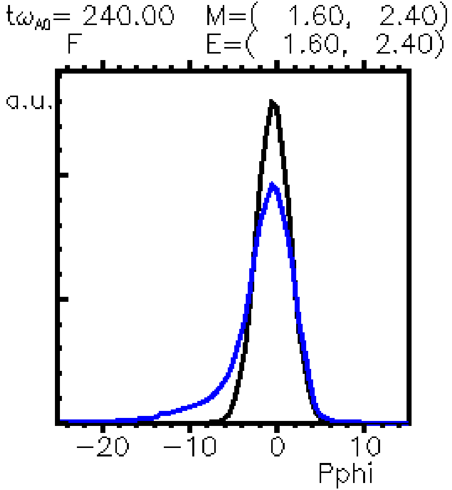
  where$(\ldots)|_\psi$ denotes orbit averaging, $\mathcal{P}_k^{-1}$ is the inverse of the operator $\mathcal{P}_k \equiv \overline{\omega}_{dk} - \omega_k - i\partial_t + i\Delta \ldots$.

- by a systematic comparison, a validated reduced model of particle and energy transport will be obtained where the essential underlying physics may be identified and understood;

# Hierarchical approach: recent developments (WP 2)

- PSZS have been extracted from an EPM simulation by the HMGC hybrid code, i.e. see Briguglio et al. 1995. Phase Space fluxes have been validated;

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# Hierarchical approach: future developments (WP 2)

- use PSZS as initial condition for a new GK simulation, e.g. with HMGC/HYMAGYK;
- solve PSZS equation in the $(\mathcal{E}, \mu, P_\phi)$ space using/developing a Fokker Planck code;
- evaluate fluxes using analytical expressions and a code solving a reduced model;
- understand how this workflow could be implemented in the European Transport Solver;

# FALCON code: theoretical framework

- In the F(loquet)AL(fvén)CON(tinuum) code, following Chen and Zonca 2016, Falessi et al. 2019a, continuous spectra are obtained from vorticity and pressure equations when $|\vartheta| \to \infty$;

$$\left(\partial_\vartheta^2 - \frac{\partial_\vartheta^2 |\nabla r|}{|\nabla r|} + \frac{\omega^2 J^2 B_0^2}{v_A^2}\right) y_1 = (2\Gamma\overline{\beta})^{1/2} \kappa_g \frac{J^2 B_0 \overline{B}_0}{q R_0} \frac{s\vartheta}{|s\vartheta|} y_2$$

$$\left(1 + \frac{c_s^2}{\omega^2} \frac{1}{J^2 B_0^2} \partial_\vartheta^2\right) y_2 = (2\Gamma\overline{\beta})^{1/2} \kappa_g \frac{\overline{B}_0}{B_0} q R_0 \frac{s\vartheta}{|s\vartheta|} y_1$$

where:

$$y_1 \equiv \frac{\hat{\phi}_s}{\left(\overline{\beta} q^2\right)^{1/2}} \frac{c k_\vartheta}{\overline{B}_0 R_0}, \quad y_2 \equiv i \frac{\delta \hat{P}_{comp}}{(2\Gamma)^{1/2} P_0},$$

- $\hat{\phi}_s(r,\vartheta) \equiv |s\vartheta| |\boldsymbol{\nabla} r| \hat{\Phi}_s(r,\vartheta)$, $s = rq'/q$ is the magnetic shear, $k_\vartheta = -nq/r$, $\overline{\beta} = 8\pi\Gamma P_0/\overline{B}_0^2$, $\hat{\Phi}_s$ and $\delta\hat{P}_{comp}(r,\vartheta)$ are the representation of the perturbed stream function and the compressional component of the pressure perturbation

# FALCON code: theoretical framework

- linear system of second order ODEs with periodic coefficients;
- Floquet theory demonstrate that it must have solutions in the form:

$$\mathbf{x}_i = e^{i\nu_i\vartheta}\mathbf{P}_i(\vartheta)$$

- where $\mathbf{P}_i$ is a $2\pi$-periodic function $i = 1, 2, 3, 4$ and the $\nu_i$ are the characteristic Floquet exponents.

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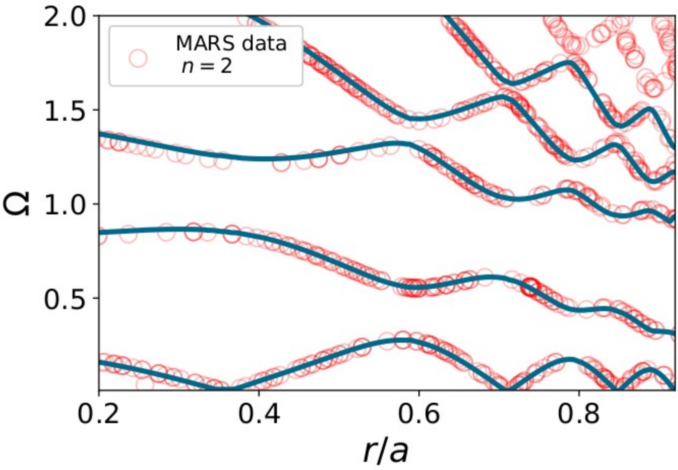
- where $\mathbf{P}_i$ is a $2\pi$-periodic function $i = 1, 2, 3, 4$ and the $\nu_i$ are the characteristic Floquet exponents.
- Solving the system for a given $r$ thus calculating $\nu_i$ for each $\omega$ we obtain:

$$\nu_i = \nu_i(\omega, r)$$

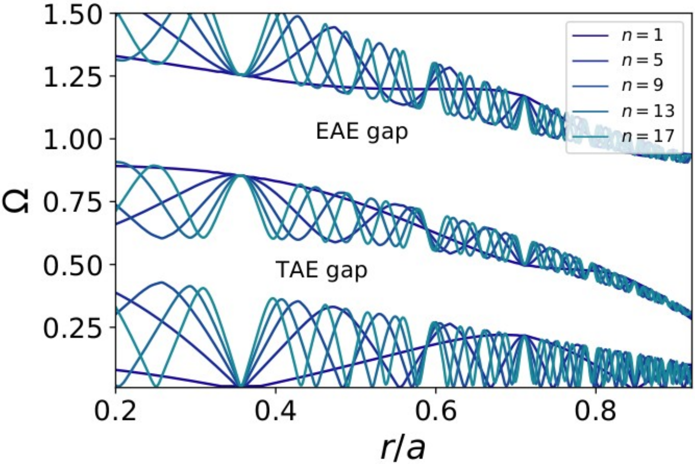
- this relation involves only local quantities and describes wave packets propagation along magnetic field lines;
- continuous spectra can be computed for arbitrary $n$, once the dispersion curves $\nu_i = \nu_i(\omega, r)$ are given;

# FALCON code: SAW continuous spectrum



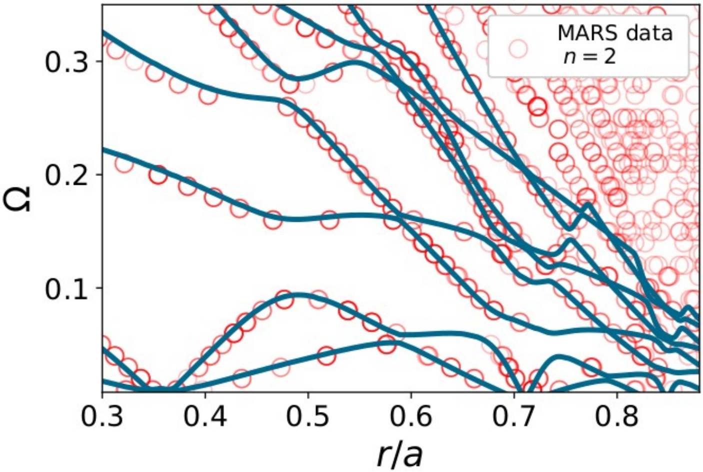

- Using $\nu_i^2(\Omega, r) = (nq(r) - m)^2$ we obtain the SAW continuous spectrum as a function of $r/a$;
- comparison with the spectrum calculated by MARS code.

# FALCON code: SAW continuous spectrum



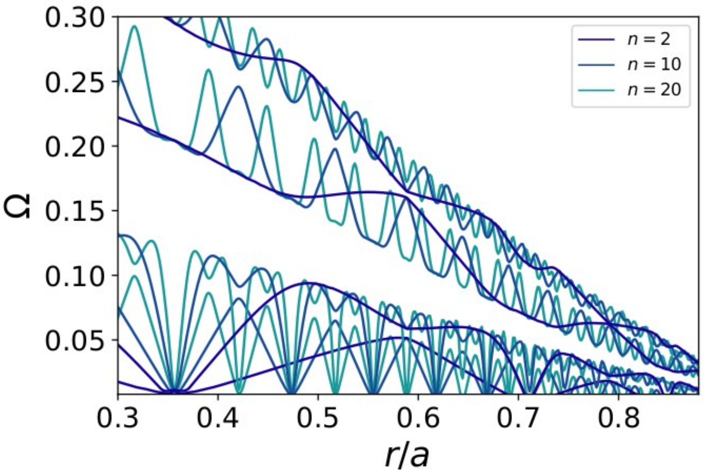

- the evaluation of the continuous spectrum for different toroidal mode numbers is immediate.

# FALCON code: benchmark with MARS code




- continuous spectra are compared with results obtained by MARS;

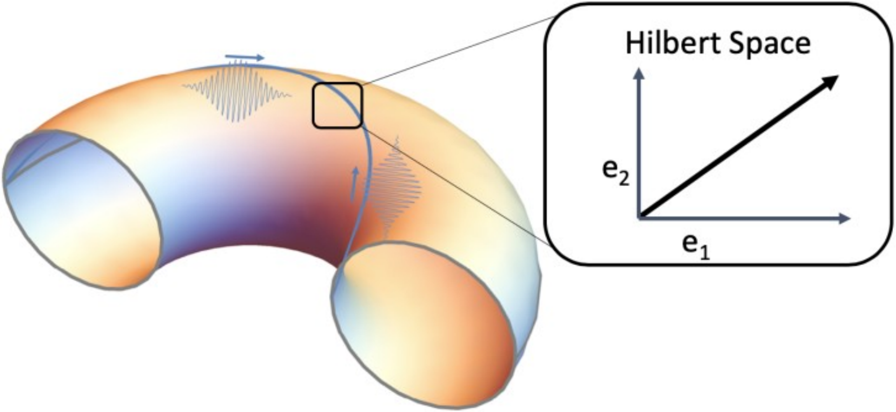
# FALCON code: SAW-ISW continuous spectrum



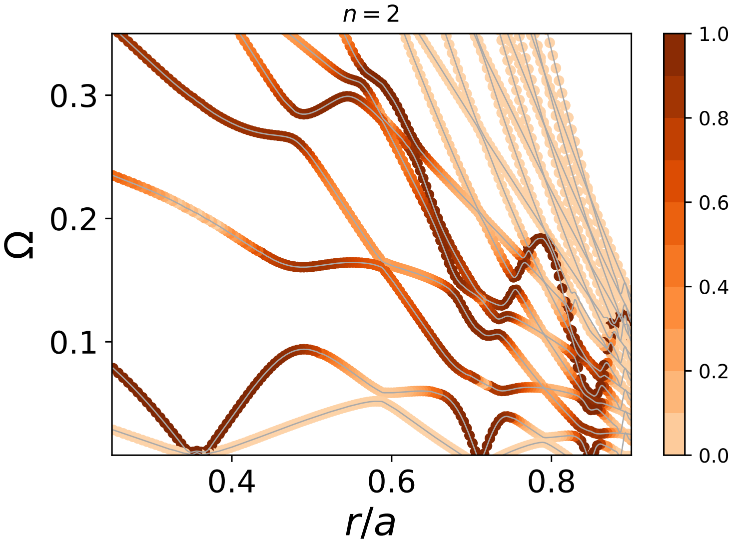

- complex structures of the continuous spectrum are further illustrated by adding $n = 10$ and $n = 20$ toroidal mode numbers;
- as stated in Chen and Zonca 2017, characterize fluctuations (in particular) in the BAAE frequency range requires to calculate polarization;

# FALCON code: polarization

- Any fluctuation (antenna driven and/or eigenmode) with short scale (large- $\eta$) radial structure satisfy the same equations;
- therefore is a linear superposition of Floquet solutions and is characterized by its own polarization;
- mode polarization is crucial in order to assess the actual coupling with the continuous spectrum and resonant absorption;

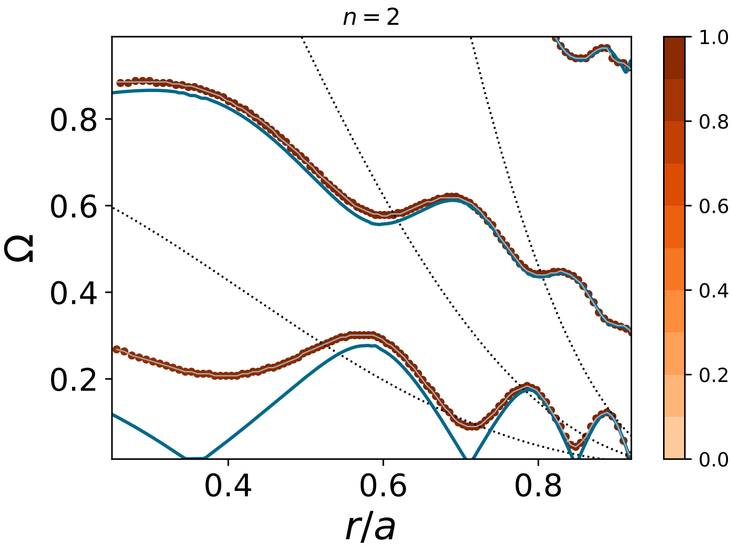


# FALCON code: recent developments (WP 2)



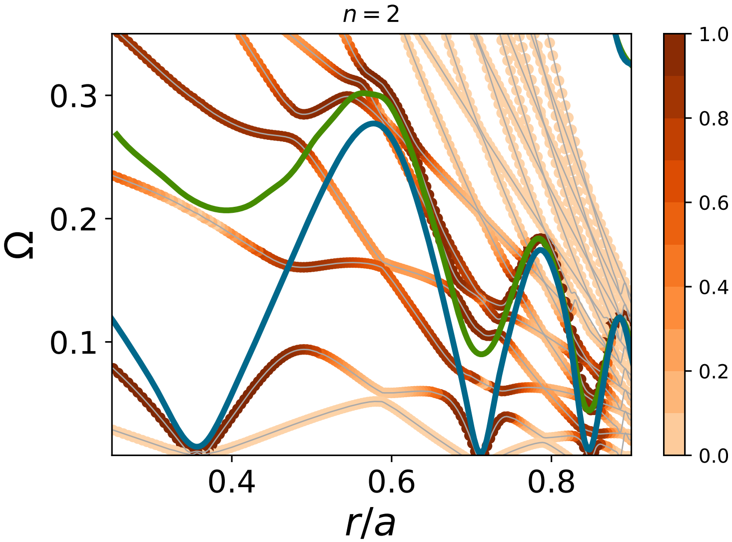

- structures near plasma edge are substantially acoustic, i.e. strongly damped;

# FALCON code: recent developments (WP 2)




- FALCON can now calculate continuous spectrum within the Slow sound approximation. Dashed line represent iso-lines of $\Pi = \hat{\rho}_{m0}/\Gamma\beta$. SAW continuous spectrum is plotted in blue;

# FALCON code: recent developments (WP 2)




- The three different level of approximation can be compared;

# FALCON code: future developments (WP 2)

- FALCON will be extended to calculate MHD parallel mode structures, i.e. the first step towards the derivation of reduced nonlinear evolution equations for Alfvénic fluctuations;
- the code is being moved to a GitLab repository which, for the moment, will be accessible only from ENEA;
- an application of the code to a DTT scenario has already been published on PoP, i.e. Falessi et al. 2019a;
- an article describing FALCON main features will be submitted to JPP in the next days;
- we are working on extending the code to include kinetic effects and 3D/stellarator geometry;

*Thank you for your attention.*