# FEL, Fractional Derivatives and Around

*Silvia Licciardi, Ph.D.*

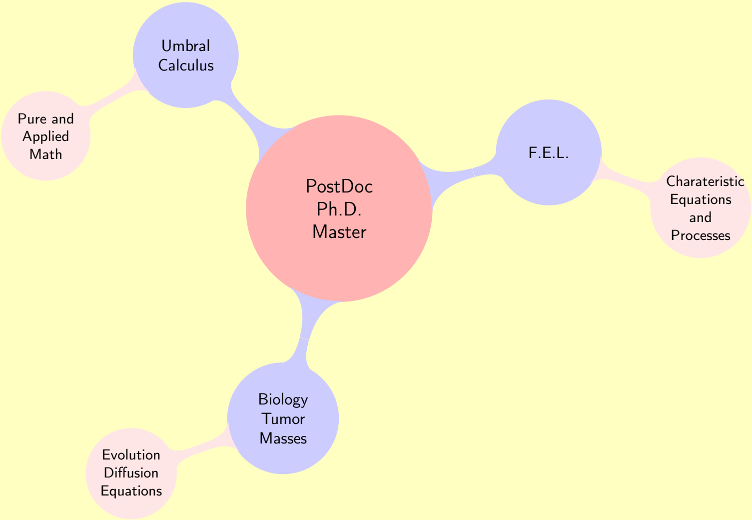
Individual Fellowship at ENEA - Frascati (Rm)

Advisors *Emanuele Di Palma, Ph.D.*
*Prof. Giuseppe Dattoli, Senior Scientist*

*Enea, Frascati (RM)*
*February 14, 2022*

Fellowship: *"Ricerca e sviluppo per la realizzazione di una sorgente FEL compatta pilotata da accelerazione al plasma"*






***"Progetto di un canale di trasporto magnetico per fasci di elettroni accelerati da laser plasma e analisi di un esperimento di tipo FEL con fasci accelerati da plasmi"***

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# Free Electron Laser

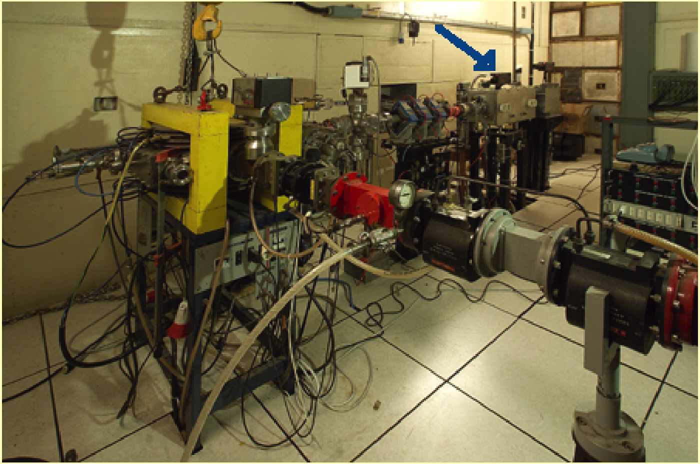

Figure: F.E.L. at Enea Frascati

# Free Electron Laser

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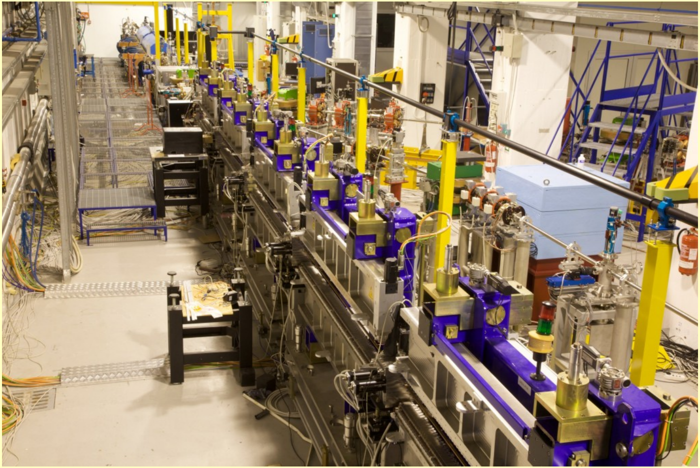

Figure: Sparc F.E.L. at INFN Frascati

# Targets: FEL and Mathematical Methods

- *Generalized Bessel Functions and their Use in Bremsstrahlung and Multi-Photon Processes*
- *Free Electron Laser high gain equation and harmonic generation*
- *Fractional Derivatives, Memory kernels and solution of Free Electron Laser Volterra type equation*

- Operator Theory, Special Functions, Fractional Calculus, Generating Function, Integral Transforms, Integro-Differential Equation...
- Further examples as *Time fractional Schroedinger-like equation*, *Pearcey Integral in Optics*, *Plasma perturbations*, etc

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# FEL Equations

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**Lienard-Wiechert $LW$ integral $\rightarrow$ spectral properties of the emitted radiation**

$$\frac{\partial^2 I}{\partial\omega\,\partial\Omega} \propto \left| \int_{-\infty}^{+\infty} \left[\vec{n} \times (\vec{n} \times \vec{\beta})\right] \exp\left[i\,\omega\,(t - \frac{\vec{n}\cdot\vec{r}}{c})\right] dt \right|^2 , \tag{1}$$

**Pendulum equation**

$$\frac{d^2}{d\,\tau^2}\,\zeta = |\,a\,|\cos\left(\zeta + \phi\right), \qquad \frac{d}{d\tau} a = -j \langle e^{-i\zeta} \rangle, \tag{2}$$

$$\tau = \frac{z}{L}, \qquad j = 2\pi g_0, \qquad a = |\,a\,|\,e^{i\phi}, \qquad \nu = \frac{d}{d\tau}\zeta$$

**Radiation growth - Volterra Integro Differential Equation**

$$\partial_\tau\, a = i\,\pi\,g_0 \int_0^\tau \tau' e^{-i\,\nu\,\tau' - \frac{(\pi\mu_\varepsilon\tau')^2}{2}} a(\tau - \tau')\,d\tau' \tag{3}$$
$$a(0) = 1$$

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**Lienard-Wiechert $LW$ integral $\rightarrow$ spectral properties of the emitted radiation**

$$\frac{\partial^2 I}{\partial\omega\,\partial\Omega} \propto \left| \int_{-\infty}^{+\infty} \left[\vec{n} \times (\vec{n} \times \vec{\beta})\right] \exp\left[i\,\omega\,(t - \frac{\vec{n}\cdot\vec{r}}{c})\right] dt \right|^2, \tag{4}$$

$I$ is the intensity, $\omega$ frequency, $\Omega$ solid angle, $\vec{n}$ observation unit vector, $\vec{r}$, $\vec{\beta} = \dot{\vec{r}}/c$ trajectory the velocity vector coordinates of the electrons [1] [2]

$$\exp\left[i\,\omega\left(t - \frac{\vec{n}\cdot\vec{r}}{c}\right)\right] \tag{5}$$

[1] G. Dattoli, E. Di Palma, S. Licciardi, E. Sabia, *Generalized Bessel Functions and Their Use in Bremsstrahlung and Multi-Photon Processes*, MDPI, Symmetry, 13, 159, 2021.

[2] Dattoli, G.; Renieri, A.; Torre, A. Lectures on Free Electron Laser and related Topics, 1st ed.;World Scientific: Singapore, 1990.

# FEL and Generalized Bessel Function



Since the electron motion assumed to be ultra-relativistic, the radiation is emitted mostly in the forward direction and therefore $\theta$ is of the order of $\frac{1}{\gamma}$ (where $\gamma >> 1$ is the electron relativistic factor). Hence the vector $\vec{n}$ can be approximated as

$$\vec{n} \equiv (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \simeq \left(\vartheta\cos(\phi), \vartheta\sin(\phi), 1 - \frac{1}{2}\vartheta^2\right) \tag{6}$$

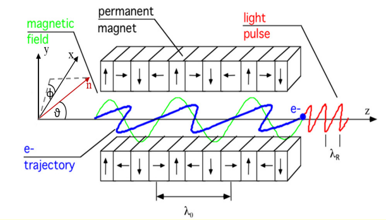


Figure: Permanent magnet block arrangement in undulators (the arrow denotes the direction of the magnetization vector), on axis field distribution and electron beam trajectory.

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Electrons equation of motion can be easily solved and at the lowest order in $1/\gamma$, thus getting[3]

$$\vec{n}\cdot\vec{r} \simeq -\frac{c}{\omega_u}\frac{K}{\gamma}\vartheta\cos(\phi)\sin(\omega_u t)+\left(1-\frac{\vartheta^2}{2}\right)\beta^* ct-\frac{K^2}{8\gamma^2}\frac{c}{\omega_u}\sin(2\,\omega_u t), \qquad (7)$$

with $\omega_u = \frac{2\,\pi\,c}{\lambda_u}, \quad \beta^* \simeq 1-\frac{1}{2\gamma^2}\left(1+\frac{K^2}{2}\right), \quad K=\frac{e\,B\,\lambda_u}{2\,\pi\,mc^2}.$

$$\exp\left[i\,\omega\left(t-\frac{\vec{n}\cdot\vec{r}}{c}\right)\right]\simeq$$

$$\simeq \exp\left[i\,\omega\left(1+\left(1-\frac{\vartheta^2}{2}\right)\beta^*\right)t\right]\exp\left[i\omega\left(A\sin(\omega_u t)+B\sin(2\,\omega_u t)\right)\right] \qquad (8)$$

where $A=-\frac{1}{\omega_u}\frac{K}{\gamma}\vartheta\cos(\phi), \quad B=-\frac{K^2}{8\,\gamma^2}\frac{1}{\omega_u}.$

[3] Dattoli, G.; Giannessi, L.; Mezi, L.; Torre, A. Theory of generalized bessel functions. Il Nuovo Cimento B (1971–1996) 1990, 105, 327–348.

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$$\exp\left[i\omega\left(A\sin(\omega_u t)+B\sin(2\,\omega_u t)\right)\right] \tag{9}$$

two variable-one index cylindrical Generalized Bessel Function by[4] $J_n(x,y)$.

$$\sum_{n=-\infty}^{+\infty} e^{i\,n\,\vartheta} J_n(x,y) = e^{i\,x\,\sin(\vartheta)+i\,y\,\sin(2\,\vartheta)}, \quad \forall x,y\in\mathbb{C} \tag{10}$$

Integral representation

$$J_n(x,y) = \frac{1}{\pi}\int_0^{\pi}\cos(n\,\vartheta - x\,\sin(\vartheta) - y\,\sin(2\,\vartheta))d\vartheta, \quad \forall x,y\in\mathbb{C}, \forall n\in\mathbb{N} \tag{11}$$

Infinite series

$$J_n(x,y) = \sum_{l=-\infty}^{+\infty} J_{n-2l}(x)\,J_l(y) \tag{12}$$

[4] It is accordingly evident that for $B \ll A \;\rightarrow\; J_n(A\omega, B\omega) \approx J_n(A\omega)$ and therefore the usual harmonic pattern is recovered.

# FEL and Generalized Bessel Function

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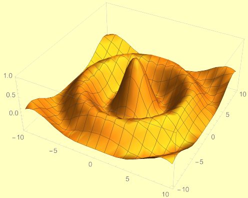
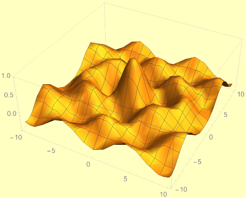
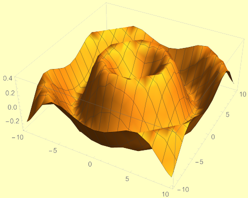


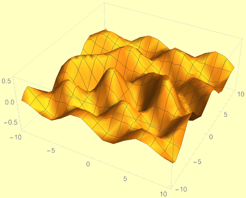

(a) $J_n(x)$ for $n = 0$.




(b) $J_n(x, y)$ for $n = 0$.




(c) $J_n(x)$for $n = 3$.




(d) $J_n(x, y)$ for $n = 3$.

Figure: Ordinary and Generalized Bessel Function for different $n$ values.

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$$\exp\left[i\,\omega\left(t-\frac{\vec{n}\cdot\vec{r}}{c}\right)\right]\simeq$$
$$\simeq\exp\left[i\,\omega\left(1+\left(1-\frac{\vartheta^2}{2}\right)\beta^*\right)t\right]\sum_{n=-\infty}^{+\infty}e^{i\,n\,\omega_u\,t}J_n(A\,\omega,\;B\,\omega). \tag{13}$$

The variables $A\omega$, $B\omega$ enter within the domain consistent with the $GBF$ correct definition, therefore there no caveats for their use within the expansion of the $LW$ exponential. After a significant amount of algebra, involving the properties of GBF

$$\frac{\partial^2 I}{\partial\omega\,\partial\Omega}\propto\sum_{m=-\infty}^{\infty}S_n\left(\frac{\omega}{\omega_1}\right)\cdot$$
$$\cdot\left[\vartheta\cos(\phi)J_m(A\omega,B\omega)+\frac{K}{2\gamma}\left(J_{m-1}(A\omega,B\omega)+J_{m+1}(A\omega,\,B\omega)\right)\right] \tag{14}$$

where $S_n(x)=\frac{2N\pi}{\omega_u}\mathrm{sinc}\left[N\,\pi\;(x-n)\right]e^{i\,N\pi\,(x-n)}$ with $\omega_1=\frac{2\pi c}{\lambda_1}$ and $\lambda_1=\frac{\lambda_u}{2\,\gamma^2}\left(1+\frac{K^2}{2}+(\gamma\,\vartheta)^2\right)$.

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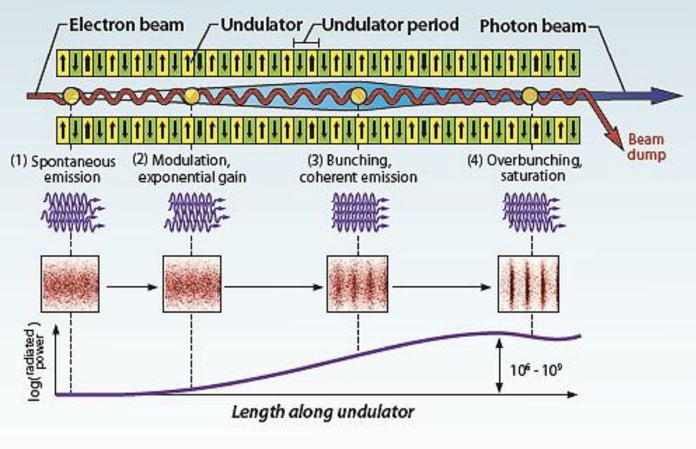


Figure: FEL in the Self Amplified Spontaneous Emission mode

# FEL Equations

## Pendulum equation

$$\frac{d^2}{d\,\tau^2}\,\zeta = |\,a\,|\cos\left(\zeta+\phi\right)\,, \qquad \frac{d}{d\tau}a = -j\langle e^{-i\zeta}\rangle, \tag{15}$$

$$\tau = \frac{z}{L}, \qquad j = 2\pi g_0, \quad a = |\,a\,|\,e^{i\phi}, \qquad \nu = \frac{d}{d\tau}\zeta$$

FEL high gain integral equation is derived from the linearization of Eqs. (15) where $z$ is the longitudinal coordinate, $L$ undulator lenght, $g_0$ small signal gain coefficient, $a$ and $j$ Colson's dimensionless amplitude and current respectively, $\zeta, \nu$ $FEL$ longitudinal phase space variables and the brackets denote average on the phase space distribution[5].

[5] G. Dattoli, E. Di Palma, S. Licciardi, E. Sabia, *Free Electron Laser High Gain Equation and Harmonic Generation*, MDPI, Appl. Sci., 11(1), 85, 2021

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Approximations $\rightarrow$ the lowest order (in the field strength) expansion of $\zeta$

$$\zeta \simeq \zeta_0 + \nu_0 \tau + \delta\zeta, \qquad \delta\zeta = \int_0^{\tau} (\tau - \tau') Re \left( a(\tau') e^{i(\zeta_0 + \nu_0 \tau')} \right) d\tau' \tag{16}$$

where $\nu_0 = 2\pi N \frac{\omega_0 - \omega}{\omega_0}$ is the detuning parameter. We take the averages on the electron-field phase $\zeta_0$ by considering the e-beam mono-energetic and without spatial and angular dispersion, thus getting, $b_n = \langle e^{-in\zeta_0} \rangle_{\zeta_0}$,

$$\langle \kappa(\zeta_0) \rangle_{\zeta_0} = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta_0) \kappa(\zeta_0) d\zeta_0, \quad f(\zeta_0) = \sum_{n=-\infty}^{\infty} b_n e^{in\zeta_0}. \tag{17}$$

and so, inserting $\delta\zeta$ in the second of Eqs. (15), we get

$$\begin{aligned} \frac{d}{d\tau} a =& -2\pi g_0 b_1 e^{-i\nu_0 \tau} + i\pi g_0 \int_0^{\tau} (\tau - \tau') a(\tau') e^{-i\nu_0(\tau - \tau')} d\tau' + \\ & + i\pi g_0 b_2 \int_0^{\tau} (\tau - \tau') a^*(\tau') e^{-i\nu_0(\tau + \tau')} d\tau' . \end{aligned} \tag{18}$$

i) Absence of an initial bunching ($f(\zeta_0)$ constant) Eq. (18)

$$\frac{d}{d\tau} a = i\pi g_0 \int_0^{\tau} (\tau - \tau') a(\tau') e^{-i\nu_0(\tau - \tau')} d\tau'. \tag{19}$$

Energy modulation and consequent bunching are due to the input coherent seed $a_0$.

# FEL, Ciocci Algorithm and Fang-Torre Formula



ii) Non constant $f(\zeta_0)$ and emergence of non zero $b_n$ coefficients, the field may grow in a seedless mode, induced by the initial bunching coefficient.

$*$) A **perturbative expansion** in the small signal gain coefficient to the 2-order in $g_0$ and neglecting[6] $b_2$: $a(\tau) \simeq a_0 + g_0 a_1 + g_0^2 a_2$

$$a_0(\tau) = 0, \;\; \frac{d}{d\tau} a_1 = -2\pi b_1 e^{-i\nu_0 \tau}, \;\; \frac{d}{d\tau} a_2 = i\pi \int_0^\tau (\tau - \tau') a_1(\tau') e^{-i\nu_0(\tau-\tau')} d\tau' \tag{20}$$

$$a(\tau) \simeq -2\pi b_1 g_0 \left( \frac{\sin\left(\frac{\nu_0 \tau}{2}\right)}{\frac{\nu_0}{2}} e^{-i\nu_0 \frac{\tau}{2}} - \frac{1}{2}\pi g_0 \frac{(i\nu_0^2 \tau^2 + 4\nu_0 \tau - 6i) e^{-i\nu_0 \tau} + 6i + 2\nu_0 \tau}{\nu_0^4} \right). \tag{21}$$

In Eq. (20) compares a double integral, eliminable by keeping two successive derivatives

$$i\pi g_0 \int_0^\tau (\tau - \tau') a(\tau') e^{-i\nu_0(\tau-\tau')} d\tau' = i\pi g_0 e^{-i\nu_0 \tau} \int_0^\tau \left( \int_0^{\tau'} a(\tau'') e^{i\nu_0 \tau''} d\tau'' \right) d\tau'. \tag{22}$$

$*$) But the $FEL$ integral equation (18) can also be reduced to the $3$-order ODE

$$\left[ \hat{D}_\tau^3 + 2i\nu_0 \hat{D}_\tau^2 - \nu_0^2 \hat{D}_\tau \right] a(\tau) = i\pi g_0 \left( a(\tau) + b_2 \; a^*(\tau) e^{-2i\nu_0 \tau} \right), \qquad \hat{D}_\tau = \frac{d}{d\tau} \tag{23}$$

and, for negligible $b_2$, Eq. (23) leads to the exponential growth also referred as $FEL$ instability[7].

[6] The $2$-order bunching coefficient does not produce any appreciable contribution and can be safely neglected.

[7] This type of instability is common to any Free Electron device (Gyrotrons and CARM too).

$a)$ Ciocci-Algorithm:

i) The solution can be written as

$$a(\tau) = e^{-i\nu_0\tau} \sum_{j=1}^{3} \kappa_j e^{-i\delta\nu_j\tau} \tag{24}$$

where $\delta\nu_j$ are the roots of the third degree algebraic equation $\delta\nu^2(\nu_0 + \delta\nu) = \pi g_0$

ii) Amplitudes $\kappa_j$ is fixed by the initial conditions $a(0) = \alpha_0 = \sum_{j=1}^{3} \kappa_j,$

$$\begin{gathered}\frac{d}{d\tau}a\Big|_{\tau=0} = -i\nu_0\alpha_0 - i\sum_{j=1}^{3}\kappa_j\delta\nu_j = -2\pi g_0 b_1 \ \rightarrow \ \sum_{j=1}^{3}\kappa_j\delta\nu_j = \alpha_1 = -\nu_0\alpha_0 - 2i\pi g_0 b_1, \\ \frac{d^2}{d\tau^2}a\Big|_{\tau=0} = -\nu_0^2\alpha_0 - 2i\nu_0\sum_{j=1}^{3}\kappa_j\delta\nu_j - \sum_{j=1}^{3}\kappa_j\left(\delta\nu_j\right)^2 \rightarrow \sum_{j=1}^{3}\kappa_j\left(\delta\nu_j\right)^2 = \alpha_2 = \nu_0^2\alpha_0 + 2i\pi\nu_0 g_0 b_1\end{gathered} \tag{25}$$

iii) Eq. (24) is cast in the more convenient form

$$a(\tau) = e^{-i\nu_0\tau}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\alpha_m\tau^m, \qquad \alpha_m = \sum_{j=1}^{3}\kappa_j\left(\delta\nu_j\right)^m. \tag{26}$$

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iv) The coefficients $\alpha_m$ are obtained from characteristic equation written as

$$\nu_0 \left(\delta v_j\right)^2 + \left(\delta \nu_j\right)^3 = \pi g_0 \tag{27}$$

which, after multiplying by $\kappa_j$ and summing on the index $j$, yields the recursion $\alpha_3 = \pi g_0 \alpha_0 - \nu_0 \alpha_2$ which is clearly generalized as (and implemented numerically)

$$\alpha_m = \pi g_0 \alpha_{m-3} - \nu_0 \alpha_{m-1}. \tag{28}$$

$b)$ Cardano's method - The Fang-Torre formula for the analytic solutions of eq. (27) can be obtained too[8] valid for $a_0 \neq 0$, $b_1 = 0$,

$$\begin{gathered} a(\tau,\nu_0) = \frac{a_0}{3(\nu_0+p+q)} e^{-\frac{2i}{3}\nu_0\tau} \left\{ (-\nu_0+p+q) e^{-\frac{i}{3}(p+q)\tau} + 2(2\nu_0+p+q) e^{\frac{i}{6}(p+q)\tau} \cdot \right. \\ \left. \cdot \left[ \cosh\left(\frac{\sqrt{3}}{6}(p-q)\tau\right) + i \frac{\sqrt{3}\nu_0}{p-q} \sinh\left(\frac{\sqrt{3}}{6}(p-q)\tau\right) \right] \right\}, \\ p = \left[\frac{1}{2}(r+\sqrt{d})\right]^{\frac{1}{3}}, \quad q = \left[\frac{1}{2}(r-\sqrt{d})\right]^{\frac{1}{3}}, \quad r = 27\pi g_0 - 2\nu_0^3, \quad d = 27\pi g_0 (27\pi g_0 - 4\nu_0^3). \end{gathered} \tag{29}$$

The $\mid a(t,\nu_0) \mid^2$ is intensity growth as a function of the dimensionless time and of the detuning parameter.

[8] This Formula appeared in an unpublished manuscript by H. Fang and later derived, with minor refinement by A. Torre, and reported in Dattoli G., Renieri A., Torre A. Lecture on Free Electron Laser Theory and related topics, World Scientific Singapore, 251 1990.

# FEL, Ciocci Algorithm and Fang-Torre Formula

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Introduction

Special Function

Cardano Method

Umbral Calculus

Fractional Calculus

Fractional PDE and ODE

Physical Applications

Further Developments

Pendulum equations including the higher order harmonics can be written as

$$\frac{d^2}{d\tau^2}\zeta = \sum_{n=0}^{\infty} \mid a_n \mid \cos(\psi_n), \qquad \psi_n = n\zeta + \phi_n, \ \frac{d}{d\tau}a_n = -j_n\langle e^{-in\zeta}\rangle, \qquad j_n = 2\pi g_{0,n} \tag{30}$$

where $a_n = \mid a_n \mid e^{i\phi_n}$ and $g_{0,n}$ are the dimensionless amplitude and harmonic small signal gain parameter of the $n$-th order harmonics

$$\frac{d}{d\tau}a_n = -2\pi g_0 b_n e^{-i\nu_n\tau} + i\pi n g_n \int_0^\tau (\tau-\tau')a_n(\tau')e^{-i\nu_n(\tau-\tau')}d\tau', \quad \nu_n = 2\pi N\frac{n\omega_0-\omega}{n\omega_0}. \tag{31}$$

Same considerations developed for the fundamental harmonic ($n=1$), hold in the present case too. The only significant difference being that the $n$-the harmonic Pierce $\rho$ is now specified by

$$\rho_n^* = \sqrt[3]{n}\,\rho_n, \qquad \rho_n = \rho\left(\frac{f_{b,n}}{f_b}\right)^{\frac{2}{3}}, \quad f_{b,n} = J_{\frac{n-1}{2}}(n\xi) - J_{\frac{n+1}{2}}(n\xi), \qquad f_{b,1} = f_b. \tag{32}$$

The gain length associated with the harmonic growth is specified by $L_{g,n} = \frac{\lambda_u}{4\pi\sqrt{3}\rho_n}$ which is larger than the corresponding value of the fundamental harmonic, namely $L^*_{g,n} = \frac{\lambda_u}{4\pi\sqrt{3}\rho^*_n} = \frac{1}{\sqrt[3]{n}}L_{g,n}$. This means that the lethargic region of small signal harmonic power growth, before the on-set of the exponential regime, is larger.

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## Radiation growth - Volterra Integro Differential Equation

$$\partial_\tau \, a = i\,\pi\, g_0 \int_0^\tau \tau' e^{-i\,\nu\,\tau' - \frac{(\pi \mu_\varepsilon \tau')^2}{2}} a(\tau - \tau')\, d\tau' \qquad (33)$$
$$a(0) = 1$$

$a$ represents the laser field amplitude, $g_0$ the small signal gain coefficient, $\nu$ is linked to the laser frequency and the coefficient $\mu_\varepsilon$ is a parameter regulating the effects of the gain reduction due to the electrons' energy distribution. It is an integro-differential equation of Volterra type[9]. The kernel of the integral part is not trivial, Eq. (34) cannot indeed been solved analytically, unless $\mu_\varepsilon = 0$.

[9] M. Artioli, G. Dattoli, S. Licciardi, S. Pagnutti; *Fractional Derivatives, Memory kernels and solution of Free Electron Laser Volterra type equation*, Mathematics 2017, 5(4), 73.

# FEL and Umbral Calculus

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$$\partial_\tau \, a = i \, \pi \, g_0 \int_0^\tau \tau' e^{-i \, \nu \, \tau' - \frac{(\pi \mu_\varepsilon \tau')^2}{2}} a(\tau - \tau') \, d\tau', \quad a(0) = 1 \tag{34}$$

## Umbral Calculus

$$f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \to \sum_{n=0}^{\infty} (\hat{c}^n \varphi_0) \frac{x^n}{n!} = e^{\hat{c}x} \varphi_0 \tag{35}$$

$$I(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} e^{-(\alpha+\beta)x^2 - \gamma x} dx = \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-(\gamma + \beta x)x} dx \tag{36}$$

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# Hermite Calculus

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Hermite Calculus[10] $\to \hat{h}\,\eta_0 \to \hat{h}$

$$e^{-(\gamma+\beta x)x} = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \mathbf{H_r}(\gamma, -\beta) \tag{37}$$

$$e^{-\hat{h}_{(\gamma,-\beta)}x} = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \hat{\mathbf{h}}^{\mathbf{r}}_{(\gamma,-\beta)} \tag{38}$$

$$I(\alpha,\beta,\gamma) = \int_{-\infty}^{\infty} e^{-\alpha x^2 - \hat{h}_{(\gamma,-\beta)}x} dx \tag{39}$$

$$\hat{h}^r_{(\gamma,-\beta)} = H_r(\gamma,-\beta) \tag{40}$$

Operator $\hat{h} \to$ *ordinary algebric quantity*

[10] Hermite Calculus, G. Dattoli, B. Germano, S. Licciardi, M.R. Martinelli, Modeling in Mathematics, Atlantis Transactions in Geometry, vol 2. pp. 43-52, 2017.

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$$e^{-\tau'\hat{h}\left(i\nu,-\frac{(\pi\mu_\varepsilon)^2}{2}\right)}=\sum_{m=0}^{\infty}\frac{(-\tau')^m}{m!}\hat{\mathbf{h}}^{\mathbf{r}}_{\left(\mathbf{i}\nu,-\frac{(\pi\mu_\varepsilon)^{\mathbf{2}}}{\mathbf{2}}\right)} \tag{41}$$

$$e^{-\left(i\nu+\frac{(\pi\mu_\varepsilon)^2}{2}\tau'\right)\tau'}=\sum_{m=0}^{\infty}\frac{(-\tau')^m}{m!}\mathbf{H_r}\left(\mathbf{i}\nu,-\frac{(\pi\mu_\varepsilon)^{\mathbf{2}}}{\mathbf{2}}\right) \tag{42}$$

$$\partial_\tau a_n=i\pi g_0\int_0^\tau \tau' e^{-\tau'\hat{h}\left(i\nu,-\frac{(\pi\mu_\varepsilon)^2}{2}\right)}a_{n-1}(\tau-\tau')d\tau'=$$

$$=i\,\pi\,g_0\sum_{m=0}^{\infty}\frac{1}{m!}\int_0^\tau \tau'^{(m+1)}H_m\left(-i\,\nu,-\frac{(\pi\,\mu_\varepsilon)^2}{2}\right)a_{n-1}(\tau-\tau')\,d\tau' \tag{43}$$

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***First order solution***

$$a_1 = i\,\pi\,g_0 \sum_{m=0}^{\infty} \frac{H_m\left(-i\,\nu, -\frac{(\pi\,\mu_\varepsilon)^2}{2}\right)}{m!} \int_0^{\tau} d\tau' \int_0^{\tau'} \tau''^{(m+1)} d\tau'', \tag{44}$$

$$\begin{aligned} a_1 &= i\,\pi\,g_0 \sum_{m_1=0}^{\infty} \alpha_{m_1} \tau^{m_1+3}, \\ \alpha_{m_1} &= \frac{H_{m_1}}{m_1!(m_1+2)\,(m_1+3)}. \end{aligned} \tag{45}$$

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***Second order solution***

$$a_2 = (i\,\pi\,g_0)^2 \sum_{m_2=0}^{\infty} \frac{H_{m_2}}{m_2!} \int_0^{\tau} d\tau' \int_0^{\tau'} \tau''^{(m_2+1)} a_1(\tau'-\tau'')d\tau'', \tag{46}$$

$$a_2 = (i\,\pi\,g_0)^2 \sum_{m_1,m_2=0}^{\infty} \alpha_{m_1,m_2} \tau^{m_1+m_2+6},$$

$$\alpha_{m_1,m_2} = \alpha_{m_1} \sum_{s=0}^{m_1+3} \begin{pmatrix} m_1+3 \\ s \end{pmatrix} \frac{H_{m_2}}{m_2!(m_2+s+2)(m_2+m_1+6)} =$$

$$= \frac{(-1)^s\; H_{m_1} H_{m_2}}{m_2!m_1!(m_1+2)\,(m_1+3)(m_2+m_1+6)} \cdot \sum_{s=0}^{m_1+3} \begin{pmatrix} m_1+3 \\ s \end{pmatrix} \frac{(-1)^s}{(m_2+s+2)}. \tag{47}$$

<u>***Higher order solution***</u>: In the present nested procedure the $n^{th}$ order can be computed in a modular way just looking at the symmetries of the expansion itself.

$$a_n = (i\,\pi\,g_0)^n \sum_{m_1,..m_n=0}^{\infty} \alpha_{m_1,..m_n} \tau^{\left(\sum_{r=1}^n m_r+3\,n\right)},$$

$$\alpha_{m_1,..m_n} = \alpha_{m_1,..m_{n-1}} \sum_{s=0}^{\left(\sum_{r=1}^{n-1} m_r+3(n-1)\right)} \begin{pmatrix} \sum_{r=1}^{n-1} m_r + 3(n-1) \\ s \end{pmatrix}. \tag{48}$$

$$\cdot \frac{(-1)^s H_{m_n}}{m_n!(m_n+s+2)\,\left(\sum_{r=1}^n m_r+3\,n\right)}.$$

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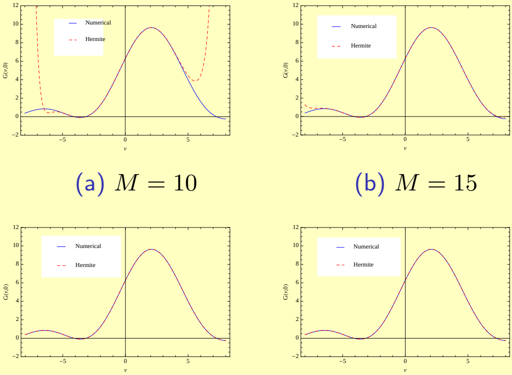
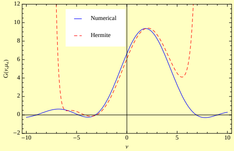



(a) $M = 10$ (b) $M = 15$

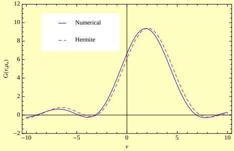
(c) $M = 20$ (d) $M = 25$

Figure: Comparison between complete numerical integration with no broadening effects ($\mu_\varepsilon = 0$) $G(\nu, 0) = \| a \|^2 - 1$ with $g_0 = 5$ at the end time $\tau = 1$, performed by Mathematica, and Hermite solution $G(\nu, 0) = \|a_0 + a_1 + a_2\|^2 - 1$ at different truncation levels .

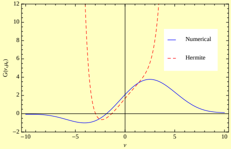
# Fel High Gain Equation



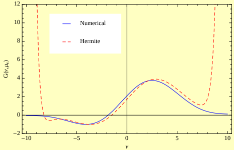

(a) $\mu_\varepsilon = 0.1, \; M = 10$




(b) $\mu_\varepsilon = 0.1, \; M = 25$




(c) $\mu_\varepsilon = 0.7, \; M = 10$




(d) $\mu_\varepsilon = 0.7, \; M = 25$

Figure: With different broadening effects $G(\nu, \mu_\varepsilon) = \| a \|^2 - 1$ with $g_0 = 5$ at the end time $\tau = 1$ and Hermite solution $G(\nu, \mu_\varepsilon) = \|a_0 + a_1 + a_2\|^2 - 1$ at different truncation levels .

# Fractional Derivatives Theory

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$$ {}_{FEL}D_{\tau} f(t) = \int_0^{\tau} \xi f(\xi) e^{-i\nu(\tau-\xi) - \frac{(\pi\mu_{\varepsilon})^2}{2}(\tau-\xi)^2} d\xi \qquad (49)$$

Real Order Fractional Derivative (Integral) according to:

- Caputo[11]

$$\hat{D}_x^{\alpha}\, g(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha} g'(\xi) d\xi\,, \quad 0 < \alpha < 1 \quad (50)$$

- Hadamard[12]

$$ {}_c\hat{D}_x^{-\alpha}\, g(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{g(\xi)}{\xi} d\xi, \quad x > c \quad (51)$$

[11] Caputo M. *Linear models of dissipation whose Q is almost frequency independent*, Geophys. J. R. astr. Soc., 13, 1967, pp. 529–539.

[12] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, J. Pure Appl. Math., 4(8), pp. 101–186, 1892.

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$$ {}_{FEL}D_{\tau} f(t) = \int_0^{\tau} \xi f(\xi) e^{-i\nu(\tau-\xi) - \frac{(\pi\mu_{\varepsilon})^2}{2}(\tau-\xi)^2} d\xi \tag{49} $$

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$$\hat{D}_x^{\alpha}\, g(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha} g'(\xi) d\xi\ , \quad 0<\alpha<1 \qquad (50)$$

- Hadamard[12]

$$ {}_c\hat{D}_x^{-\alpha}\, g(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \left(\log\frac{x}{\xi}\right)^{\alpha-1} \frac{g(\xi)}{\xi} d\xi, \quad x>c \qquad (51)$$

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[11] Caputo M. *Linear models of dissipation whose Q is almost frequency independent*, Geophys. J. R. astr. Soc., 13, 1967, pp. 529–539.

[12] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, J. Pure Appl. Math., 4(8), pp. 101–186, 1892.

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Real Order Fractional Derivative (Integral) according to:

- Atangana-Baleanu[13]

$$ {}^{f}_{c}\hat{D}_x^{-\alpha}\, g(x) = \frac{1-\alpha}{f(\alpha)} g(x) + \frac{\alpha}{f(\alpha)\Gamma(\alpha)} \int_c^x (x-\xi)^{\alpha-1} g(\xi)d\xi \tag{52} $$

with $f(\alpha)$ function of normalization : $f(0) = f(1) = 1$. It exploits a generalized Mittag-Leffler function and with a kernel not local and not singular.

- Euler-Riemann-Liouville[14]

$$ {}_{c}\hat{D}_x^{-\alpha}\, g(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-\xi)^{\alpha-1} g(\xi)d\xi, \quad \alpha > 0 \tag{53} $$

with $g(x)$ is piecewise continuos on $(0,\infty)$ and integrable on any finite subinterval based on the Lagrange rule for differential operators

[13] A. Atangana, D. Baleanu New fractional derivatives with non-local and non-singular kernel: theory and applications to heat transfer model, Therm Sci, 20 (2016), pp. 763-769

[14] K.B. Oldham, J. Spanier; *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Mathematics in Science and Engineering, vol 111, 1974.

## Corollary

Using the ***Euler-Riemann-Liouville*** definition for real order derivative

$$\hat{D}_x^{\alpha} x^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} x^{\nu-\alpha}, \quad \forall x, \alpha, \nu \in \mathbb{R}, \tag{54}$$

we find[a]

$$\hat{D}_x^{\alpha}\, E_{\alpha,1}(\lambda\, x^{\alpha}) = \lambda\, E_{\alpha,1}(\lambda x^{\alpha}) + \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad \forall x, \lambda \in \mathbb{R}, \forall \alpha \in \mathbb{R}^{+}. \tag{55}$$

[a] The extra-term emerges because, according to Eq. (54), the fractional derivative of a constant does not vanish.

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We note that $E_{n,1}(\lambda x^n)$ is an eigenfunction of the $\partial_x^n$ operator $\forall x, \lambda \in \mathbb{R}$, $\forall n \in \mathbb{N}$, therefore

**Lemma**

$$\partial_x^n \, E_{n,1}(\lambda x^n) = \lambda E_{n,1}(\lambda x^n), \quad \forall n \in \mathbb{N}, \forall x, \lambda \in \mathbb{R}. \tag{56}$$

It can be extended also to the case of real order *ML* functions. In this case **derivatives of non-integer order** have to be considered.

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# Fractional Evolution Problem

## Example

$\forall x \in \mathbb{R}, \forall \alpha \in \mathbb{R}^+, \forall t \in \mathbb{R}_0^+$,

$$\begin{cases} \partial_t^\alpha F(x,t) = \partial_x^2 \, F(x,t) + \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} f(x), \\ F(x,0) = f(x), \end{cases} \tag{57}$$

defines a ***time-fractional diffusive*** *equation*.

According to the previous discussion, to the fact that the *ML* "$E_{\alpha,1}(t^\alpha)$" is an eigenfunction of the fractional derivative operator, according to the definition (55) and considering the formalism developed, we can obtain the relevant solution in the form

## Solution

$$F(x,t) = E_{\alpha,1}(t^\alpha \partial_x^2) \, f(x) \tag{58}$$

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# Pseudo-Evolution Operator

## Definition

$\forall \alpha \in \mathbb{R}^+, \forall t \in \mathbb{R}_0^+$, $\mathbf{E}_{\alpha,1}(\mathbf{t}^\alpha \partial_{\mathbf{x}}^{\mathbf{2}})$ is the **pseudo-evolution operator** .

The relevant action on the initial function can be espressed as

$$F(x,t) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{+\infty} E_{\alpha,1}(-t^\alpha k^2)\, \tilde{f}(k)\, e^{i\,x\,k} dk, \tag{59}$$

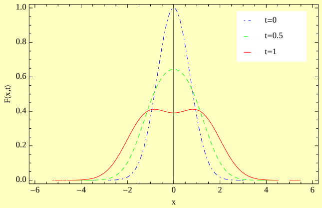
where $\tilde{f}(k)$ is the Fourier transform of $f(x)$.

**Proof.**

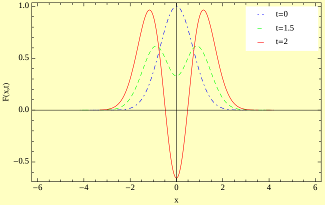
$$F(x,t) = E_{\alpha,1}(t^\alpha \partial_x^2)\, f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_{\alpha,1}(t^\alpha\ \partial_x^2) \tilde{f}(k) e^{ixk} dk = \frac{1}{\sqrt{2\pi}} \cdot$$
$$\cdot \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{t^{\alpha r} \partial_x^{2r}}{\Gamma(\alpha r + 1)} \tilde{f}(k) e^{ixk} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{t^{\alpha r}}{\Gamma(\alpha r+1)} (ik)^{2r} \tilde{f}(k) e^{ixk} dk = \tag{60}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{t^{\alpha r}(-k^2)^r}{\Gamma(\alpha r+1)} \tilde{f}(k) e^{ixk} dk$$

# Time-Fractional Diffusive Equation Solutions






(a) $\alpha = 1.5$.




(b) $\alpha = 3.5$.

Figure: Solution $F(x,t)$ for $f(x) = e^{-x^2} \to \tilde{f}(k) = \dfrac{e^{-\frac{k^2}{4}}}{\sqrt{2}}$, at different times for different $\alpha$ values.

Behaviour not simply diffusive but also ***anomalous***. Important role in the description of processes called *super* or *sub-diffusive*.

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We solve the ***time fractional Schrödinger equation*** (*FSE*) for a physical process implying the emission and absorption of photons. We assume that the relevant dynamics is ruled by the *ML* - Schrödinger equation [15]

$$\begin{aligned} & i^\alpha\, \partial_t^\alpha \mid \Psi \rangle = \hat{H} \mid \Psi \rangle + i^\alpha \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \mid \Psi(0) \rangle, \\ & \hat{H} = i^\alpha\, \Omega \left(\hat{a} - \hat{a}^+\right), \quad 0 \leq \alpha \leq 1, \forall t \in \mathbb{R}_0^+ \end{aligned} \tag{61}$$

where $\hat{a}, \hat{a}^+$ are annihilation, creation operators satisfying the commutation relation $[\hat{a}, \hat{a}^+] = \hat{1}$ and the constant $\Omega$ has the dimension of $t^{-\alpha}$.

If we work in a *Fock* basis and choose the "physical" vacuum (namely the state of the quantized electromagnetic field with no photons) as the initial state of our process namely $\mid \Psi(t) \rangle \mid_{t=0} = \mid 0 \rangle$, we can understand how the field ruled by a $FSE$ evolves from the vacuum.

[15] According to Dirac notation we write the state $\mid \Psi \rangle$ to indicate the function $\Psi(t)$.

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The solution of our *FSE* therefore writes

$$\begin{aligned} \mid \Psi \rangle &= e^{\,{}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\, \left(\hat{a}-\hat{a}^{+}\right)} \mid 0 \rangle = \\ &= e^{-\frac{\left({}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\right)^{2}}{2}} e^{-\left({}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\right) \hat{a}^{+}} e^{\left({}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\right) \hat{a}} \mid 0 \rangle. \end{aligned} \tag{62}$$

The use of the identities $(\hat{a}^{+})^{n} \mid 0 \rangle = \sqrt{n!} \mid n \rangle$ and $\hat{a} \mid 0 \rangle = 0$ finally yields the solution in the form

$$\begin{aligned} \mid \Psi \rangle &= e^{-\frac{\left({}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\right)^{2}}{2}} e^{-\left({}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\right)\hat{a}^{+}} \mid 0 \rangle = \\ &= e^{-\frac{\left({}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\right)^{2}}{2}} \sum_{n=0}^{\infty} \frac{\left(-\,{}_{\alpha}\hat{d}\, t^{\alpha}\, \Omega\right)^{n}}{\sqrt{n!}} \mid n \rangle. \end{aligned} \tag{63}$$

# Examples

$\forall \alpha, \beta \in \mathbb{R}^+$

$$I_{\alpha,\,\beta} = \int_{-\infty}^{\infty} E_{\alpha,\beta}(-x^2)dx \tag{64}$$

1. $I_{\alpha,\,\beta} = \hat{c}^{\beta-1}\int_{-\infty}^{\infty} \frac{1}{1+\hat{c}^{\,\alpha} x^2}dx\,\varphi_0 =$
   $\hat{c}^{\beta-1}\int_{-\infty}^{\infty} \frac{1}{1+\hat{c}^{\,\alpha} x^2}dx\,\varphi_0 = \pi\,\hat{c}^{\beta-\frac{\alpha}{2}-1}\varphi_0 = \frac{\pi}{\Gamma\left(\beta-\frac{\alpha}{2}\right)}$
2. $I_{\alpha,\,\beta} = \left(\sqrt{\pi}\hat{c}^{\beta-\frac{\alpha}{2}-1}\int_0^{\infty} e^{-s}s^{-\frac{1}{2}}ds\right)\varphi_0 =$
   $\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)\hat{c}^{\beta-\frac{\alpha}{2}-1}\varphi_0 = \frac{\pi}{\Gamma\left(\beta-\frac{\alpha}{2}\right)}$
3. $I_{\alpha,\beta} = \int_{-\infty}^{\infty} E_{\alpha,\beta}(-x^2)dx = \int_{-\infty}^{\infty} e^{-{}_{\alpha,\beta}\hat{d}\,x^2}dx\,\psi_0 =$
   $\sqrt{\pi}\left({}_{\alpha,\,\beta}\hat{d}\right)^{-\frac{1}{2}}\psi_0 = \frac{\pi}{\Gamma\left(\beta-\frac{\alpha}{2}\right)}$

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1) *Non-perturbative treatment of the anharmonic oscillator*[16]

$$\mathbf{J(a,b,c)} = \int_{-\infty}^{\infty} \mathbf{e}^{-(\mathbf{ax^4+bx^2+cx})}\mathbf{dx}, \quad Re(a) > 0 \tag{65}$$

$$J(a,b,c) = \int_{-\infty}^{\infty} e^{-\hat{h}(b,-a)x^2 - cx} dx = \sqrt{\frac{\pi}{\hat{h}}} e^{\frac{c^2}{4\hat{h}}} = \\ = \sqrt{\pi} \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{c}{2}\right)^{2s} \hat{\mathbf{h}}^{-\left(\mathbf{s}+\frac{1}{2}\right)} \tag{66}$$

$$\hat{h}^{-\left(s+\frac{1}{2}\right)} = H_{-\left(s+\frac{1}{2}\right)}(b,-a) \tag{67}$$

$$I(\gamma,\beta) = \int_{-\infty}^{\infty} e^{-\hat{h}x^2} dx = \sqrt{\pi}\hat{h}^{-\frac{1}{2}} \tag{68}$$

2) *Pearcey Integral* used in Optics for diffraction problems[17]

$$J(1,x,-iy) = \int_{-\infty}^{\infty} e^{-(t^4+xt^2)+iyt} dt = \sqrt{\frac{\pi}{\hat{h}_{(x,-1)}}} e^{-\frac{y^2}{4\hat{h}_{(x,-1)}}} \tag{69}$$

$$J(1,x,-iy) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} g_n(1) \left[H_n(-iy,-x) + H_n(iy,-x)\right] \tag{70}$$

[16] J. Bohacik, P. Augustin and P. Presnajder, *"Non-perturbative anharmonic correction to Mehler's presentation of the harmonic oscillator propagator"*, Ukr. J. Phys. 59, 179 (2014)

[17] Jos L. Lopez and Pedro J. Pagola, arXiv:1601.03615 [mat.NA].

# Bessel Functions

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$$\mathbf{J_0(x) = e^{-\hat{c}\left(\frac{x}{2}\right)^2}\varphi_0} \quad {}^{1} \tag{71}$$

$$\int_0^\infty J_0(x)dx = \left(\int_0^\infty e^{-\hat{c}\left(\frac{x}{2}\right)^2}dx\right)\varphi_0 = \sqrt{\pi}\frac{1}{\Gamma\left(\frac{1}{2}\right)} = 1. \tag{72}$$

$$f(x;a,b) := J_0(ax)J_0(bx), \tag{73}$$

$$\mathbf{f(x;a,b) = e^{-(a^2\hat{c}_1+b^2\hat{c}_2)\left(\frac{x}{2}\right)^2}\varphi_0^{(1)}\varphi_0^{(2)}}, \tag{74}$$

$$\begin{aligned} f(x;a,b) &= \sum_{r=0}^\infty \frac{(-1)^r}{r!} l_r(a^2,b^2)\left(\frac{x}{2}\right)^{2r}, \\ l_r(a,b) &= r!\sum_{s=0}^r \frac{a^{(r-s)}b^s}{(s!)^2\left[(r-s)!\right]^2}. \end{aligned} \tag{75}$$

[1] G. Dattoli, E. Sabia, E. Di Palma, S. Licciardi; "Products of Bessel functions and associated polynomials"; Applied Math. and Comp., Vol 266 Issue C, 2015, pp 507-514, Elsevier Science Inc. New York, NY, USA

# Plasma perturbation

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Equation descrabing the evolution of magnetized turbolent plasma

$$\gamma \partial_t^2 \phi = \partial_t \nabla^2 \phi + \alpha(\partial_x \phi \partial_y - \partial_y \phi \partial_x)\nabla^2 \phi \tag{76}$$

where $\phi$ is the electric field scalar potential with initial condition

$$\phi(x,y,0) = 0.01 + \frac{1}{1000} rnd(0,1), \quad \partial_t \phi(x,y,t)\,|_{t=0} = 0.1,$$
$$\gamma = 0.0188, \ \alpha = 1.$$

Mathematica - Fipy (object oriented, PDE solver, written in Python)

$$\begin{cases} \partial_t \phi = \psi_1 \\ \partial_t \psi_2 = \nabla^2 \psi_1 \\ \partial_t(\gamma \psi_1) = \nabla^2 \psi_1 + \alpha \partial_i(\phi \Gamma_{ij} \partial_j \psi_2), \quad \Gamma_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{cases} \tag{77}$$

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Mathematica: {Eq. (76) Explicit RK, Eq. (77) Perturbative approach}
Fipy : {Eq. (77) standard finite volume}
Neumann Boundary conditions ($=0$) and different examples of initial conditions:

$$
\begin{gathered}
1)\ \phi(x,y,0)=\frac{1}{100}J_0\left(6\sqrt{(x-1)^2+(y-1)^2}\right)\\
2)\left\{\begin{array}{ll}\phi(x,y,0)=1 & x,y>1\\ \phi(x,y,0)=0 & x,y\geq 1\end{array}\right. \qquad 3)\ \phi(x,y,0)=\frac{1}{10}J_3(10x,10y),\\
4)\ \phi(x,y,0)=\frac{1}{100}\sin(10x),\\
5)\ \phi(x,y,0)=0.01+\frac{1}{1000}rnd(()(2-1)).
\end{gathered}
\tag{78}
$$

# Plasma perturbations

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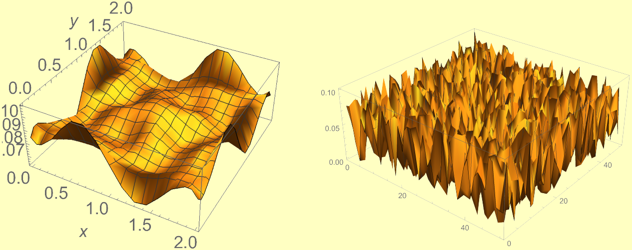


Solution $\phi$ of Eq. (76) with i.c. $\frac{1}{100}J_0(6\sqrt{(x-1)^2+(y-1)^2})$ and $0.01+\dfrac{1}{1000}rnd(()(2-1))$, $\alpha=1$, Neumann b.c. $=0$ and $\partial_t\phi(x,y,t)\mid_{t=0}=0.1$.

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Thanks to all

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