On an analytical optimization of plasma density profiles for downramp injection in LWFA - CIP2026

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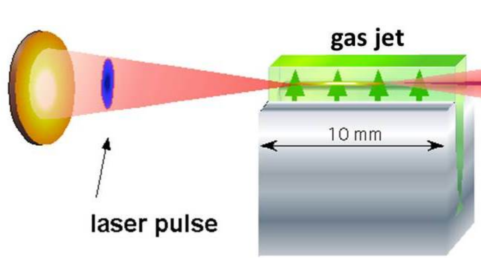
Abstract: We sketch a multi-step preliminary analytical procedure [1,2] that tailors the initial density $\widetilde{n_0}(z)$ (upramp, downramp, plateau) of a cold diluted collisionless plasma to a very short and intense laser pulse travelling in the z direction, so as to maximize the early laser wakefield acceleration (LWFA) of bunches of plasma electrons self-injected in the plasma wave (PW) by the 1st wave-breaking (WB) at the density downramp. Applying it to a Gaussian laser pulse with $l_{fwhm} = 10.5\lambda$, $a_0 = 2$ and two associated $\widetilde{n_0}(Z)$, we then determine the detailed plasma dynamics by FB-PIC simulations, confirming the predicted maximal acceleration in the early LWFA stages.

I. Introduction and set-up

Laser wake-field acceleration (LWFA) [Tajima, Dawson '79]: seminal mechanism of extreme acceleration: electrons (e^-s) "surf" a plasma wave (PW) driven by a very short laser pulse. Dynamics ruled by Maxwell eqs + kinetic theory eqs for plasma e^-s, ions.



Direct Problem (DP): solve these es equipped with "input data", i.e. initial electromagnetic (EM) field, densities and velocities of the plasma components, to get as "output" the evolution of the EM field and motion of the plasma. Today DPs can be solved via very powerful but costly particle-in-cell (PIC) simulations.
Inverse Problem (IP): given the desired output (accelerated electron bunches of high quality,...) find an input generating it.
In general this is an unsolved and formidable task!
Approaching such an input by trial and error via PIC simulations only is unaffordable. Better: run them to fine-tune inputs preliminarly selected via intuition & simpler models. Even better: run them after solving the IP, at least approximately and in simplified situations. We do this in [1,2] via a mainly **plane** analytic model.

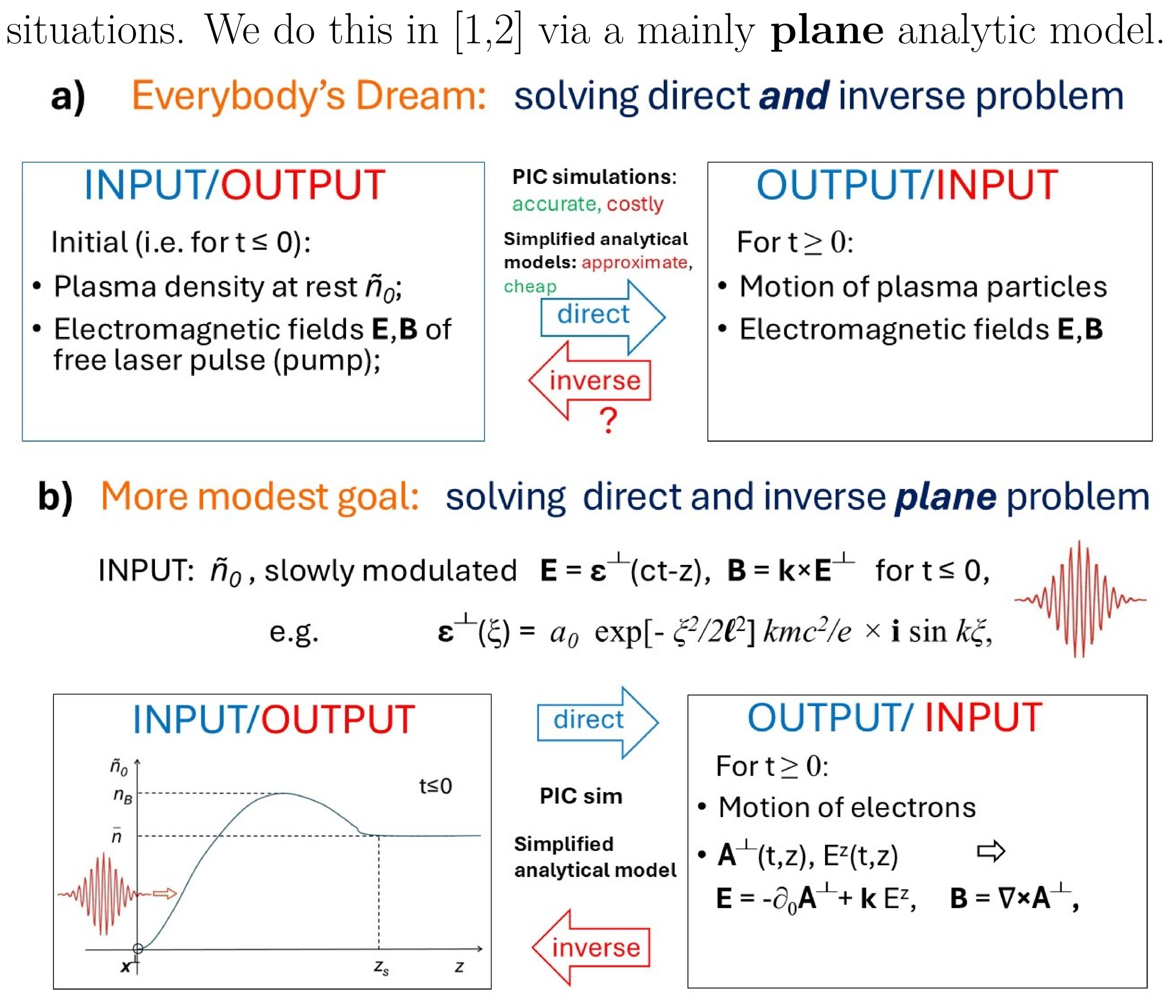


Figure 1:Direct and inverse problems

We regard the plasma as collisionless, initially at rest, with a static ion background. As $|\mathbf{v}|<c$, we can simplify the Lorentz eqs. for a e^- changing independent parameter $t \mapsto \xi = ct-z$ along its worldline (WL), see Fig. 2: this transforms the electric force $-e\boldsymbol{\epsilon}^\perp[ct-z(t)]$, containing the *unknown* $z(t)$, into the *known* forcing term $-e\boldsymbol{\epsilon}^\perp(\xi)$.

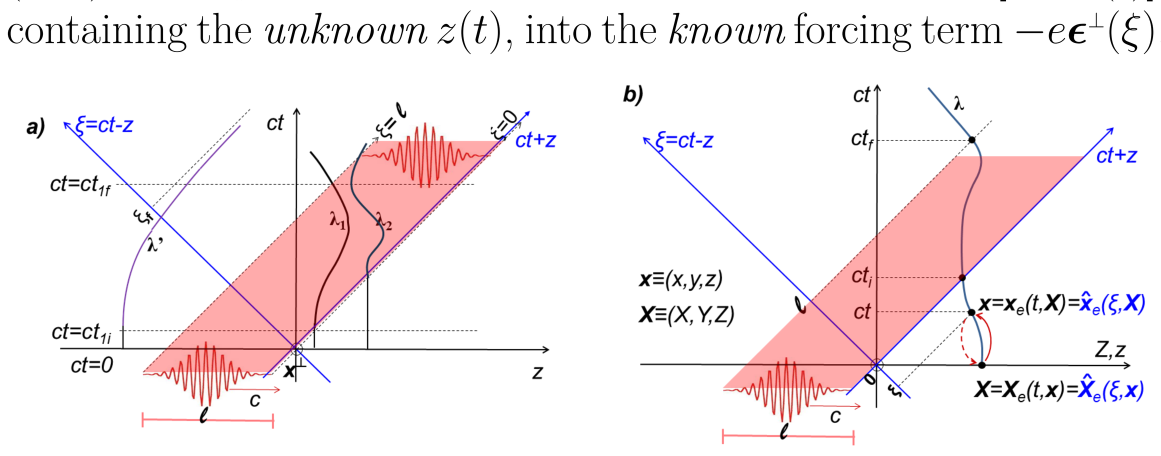


Figure 2: ξ vs. t as a parameter on the WL of: a) a single e^-; b) a plasma e^-. Front, end of the EM wave (pink) intersect different WLs at different t-instants ($t_{1i} \neq t_{2i}$, $t_{1f} \neq t_{2f}$), but at the same ξ-instants $\xi_i = 0$, $\xi_f = l$. WL λ' has an asymptote $\xi = \xi_f < \infty$: $\xi \to \xi_f$ amounts to $t \to \infty$ with $\dot{z} \xrightarrow{t\to\infty} c$. We label the e^- fluid elements by their initial positions $\mathbf{X}$; the hydrodynamic regime (HR) holds as long as their WLs do not cross, i.e. $\mathbf{X} \mapsto \mathbf{x}$ are 1-to-1 maps. Eulerian observables $f(t,\mathbf{x}) = \check{f}(\xi,\mathbf{x}) = \tilde{f}(t,\mathbf{X}) = \hat{f}(\xi,\mathbf{X})$ Lagrangian observables. We dub as 'Z electrons' the e^-s in the layer $[Z, Z+dZ]$ for $t \leq 0$.

$\mathbf{x}, \mathbf{p} = mc\mathbf{u} \equiv$ position, momentum of e^-. Dimensionless variables: $\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} = \frac{\dot{\mathbf{x}}}{c}$, $\gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}} = \sqrt{1+\mathbf{u}^2}$, 4-velocity $u = (u^0, \mathbf{u}) \equiv (\gamma, \gamma\boldsymbol{\beta})$,

$$s \equiv \gamma - u^z = \gamma(1-\beta^z) = \frac{\gamma}{c}\frac{d\tilde{\xi}}{dt} > 0; \tag{1}$$

$s \equiv$ lightlike component of u. $\gamma, \mathbf{u}, \boldsymbol{\beta}$ are functions of $\mathbf{u}^\perp, s$:

$$\gamma = \frac{1+\mathbf{u}^{\perp 2}+s^2}{2s}, \quad u^z = \frac{1+\mathbf{u}^{\perp 2}-s^2}{2s}, \quad \boldsymbol{\beta} = \frac{\mathbf{u}}{\gamma}. \tag{2}$$

Observables $f(t)$ as functions of ξ: $\hat{f}(\xi) \equiv f(t)$; let $\hat{f}' \equiv d\hat{f}/d\xi$; $c\hat{t}(\xi) \equiv \xi + \hat{z}(\xi)$. If $\hat{s}(\xi) \to 0$ as $\xi \uparrow \xi_f < \infty$, then $\hat{\gamma}, \hat{u}^z, \hat{t} \to \infty$.
As the Eulerian e^- momentum $\mathbf{p}_e$ satisfies $\mathbf{p}_e^\perp(0,\mathbf{x}) = \mathbf{0}$, we find

$$\mathbf{p}_e^\perp = \frac{e}{c}\mathbf{A}^\perp \qquad \text{i.e.} \quad \mathbf{u}_e^\perp = \frac{e}{mc^2}\mathbf{A}^\perp, \tag{3}$$

$\mathbf{A}^\perp(t,z) \equiv -c\int_{-\infty}^{t} \mathbf{E}^\perp(t',z)dt'$: we shall trade $\mathbf{u}_e^\perp$ for $\mathbf{A}^\perp$ as unknowns.

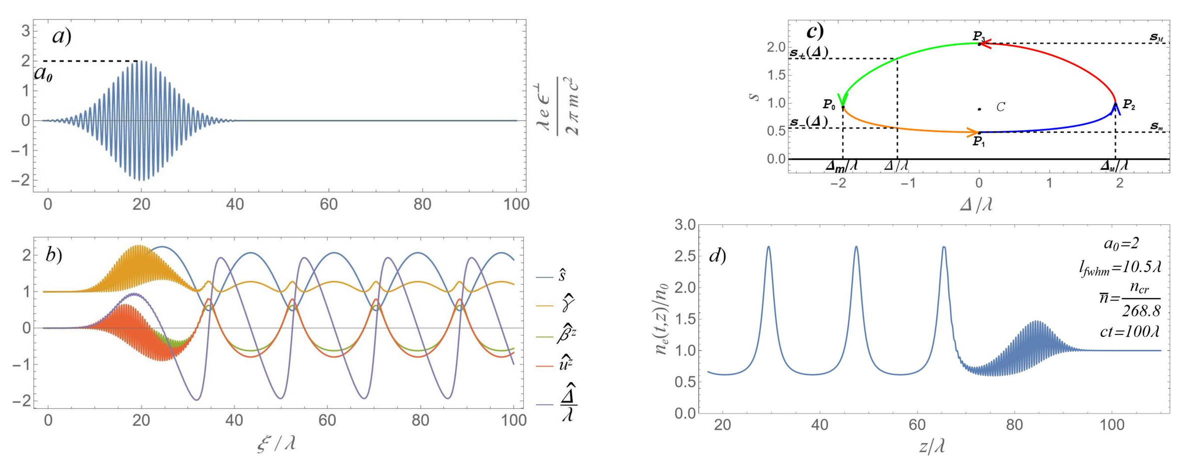


Figure 3:(a) Normalized SMM pulse with Gaussian envelope of FWHM $l' = 10.5\lambda$, linear polarization, peak amplitude $a_0 \equiv \lambda e E_M^\perp / 2\pi mc^2 = 2$. If $\widetilde{n_0}(z) = \bar{n}^j \equiv n_{cr}/269$ ($n_{cr} = \pi mc^2/e^2\lambda^2 =$), then $E/mc^2 \equiv h = 1.28$, and corresponding: (b) solution of (7-8); (c) electron phase portrait (at $\xi > l$); (d) electron density at $t = 100\lambda/c$. If $\lambda = 0.8\mu$m, then peak intensity is $I = 1.7\times 10^{19}$W/cm^2 and $\bar{n}^j = 6.5\times 10^{18}$cm^{-3}.

$$N_e(t,z) \equiv \int_0^\infty dZ\, \widetilde{n_0}(Z)\, \theta\Big[z - z_e(t,Z)\Big] \tag{4}$$

($\theta \equiv$ Heaviside step function) is the number of e^- per unit transverse surface having coordinate $z' \leq z$ at time t. The Eulerian electron density is $n_e = \partial_z N_e$. Maxwell eqs $\partial_t E^z/c + 4\pi j^z = (\nabla\wedge\mathbf{B})^z = 0$, $\nabla\cdot\mathbf{E} - 4\pi j^0 = \partial_z E^z - 4\pi e(n_p - n_e) = 0$ are solved by [5]

$$E^z(t,z) = 4\pi e\Big[\widetilde{N}(z) - N_e(t,z)\Big] \tag{5}$$

where $\mathbf{j} = -en_e\boldsymbol{\beta}_e$, $\widetilde{N}(z) \equiv \int_0^z d\eta\, \widetilde{n_0}(\eta)$; thereby they are regularized where $n_e(t,z)$ diverges due to WBs: $N_e(t,z)$ keeps finite also there. Via $n_e = \partial_z N_e$, (5) we express n_e, E^z through $\widetilde{n_0}$ and the still unknown $z_e(t,Z)$. $\hat{\mathbf{x}}_e(\xi,\mathbf{X}) - \mathbf{X}$ depends only on ξ, Z and vanishes if $\xi \leq 0$. $\hat{\mathbf{x}}_e^\perp$ is obtained integrating $\hat{\mathbf{x}}_e^{\perp\prime} = \hat{\mathbf{u}}_e^\perp/\hat{s}$, following from (2c). The remaining unknowns $\mathbf{A}^\perp$, $\hat{\Delta}(\xi,Z) \equiv \hat{z}_e(\xi,Z) - Z$, $\hat{s}$ satisfy

$$\left(\tfrac{1}{c^2}\partial_t^2 - \partial_z^2\right)\mathbf{A}^\perp = -\mathbf{A}^\perp K\gamma_e/n_e, \tag{6}$$

and $\hat{\Delta}' = \hat{u}_e^z/\hat{s}$, $\hat{s}' = \frac{e}{mc^2}\check{E}^z(\xi, \hat{z}_e)$, which *in the HR* become

$$\hat{\Delta}' = \frac{1+v}{2\hat{s}^2} - \frac{1}{2}, \quad \hat{s}' = K\Big\{\widetilde{N}\big[Z+\hat{\Delta}\big] - \widetilde{N}(Z)\Big\}, \tag{7}$$

(here $K \equiv \frac{4\pi e^2}{mc^2}$, $v \equiv \hat{\mathbf{u}}_e^{\perp 2} = \left[\frac{e\hat{\mathbf{A}}^\perp}{mc^2}\right]^2$), with 'initial conditions'

$$\hat{\Delta}(0,Z) = 0, \quad \hat{s}(0,Z) = 1, \quad \mathbf{A}^\perp(t,z) = \boldsymbol{\alpha}^\perp(ct-z) \ \text{ for } t \leq 0, \tag{8}$$

cf. Fig. 1b; here $\boldsymbol{\alpha}^\perp(\xi) \equiv -\int_{-\infty}^{\xi} \boldsymbol{\epsilon}^\perp(\eta)d\eta$. *In general*, (5b) gives E^z,

$$\hat{s}'(\xi,Z) = K\left\{\widetilde{N}\big[\hat{z}_e(\xi,Z)\big] - \int_0^\infty d\zeta\, \widetilde{n_0}(\zeta)\, \theta\big[\hat{z}_e(\xi,Z) - \hat{z}_e(\xi,\zeta)\big]\right\} \tag{9}$$

replaces (7b); the 1st, 2nd terms at the rhs are due to the interaction of the Z e^-s resp. with ions, other e^-s. Eq (6) is Maxwell eq $\Box\mathbf{A}^\perp = 4\pi\mathbf{j}^\perp$. Eq. (8c) approximately holds also for small $t > 0$, making (7) a Z-family of *decoupled Hamilton eqs* $\hat{\Delta}' = -\partial\hat{H}/\partial\hat{s}$, $\hat{s}' = \partial\hat{H}/\partial\hat{\Delta}$ of a 1D system: $\xi, \hat{\Delta}, -\hat{s}$ play the role of t, q, p,

$$\hat{H}(\hat{\Delta}, \hat{s}, \xi; Z) \equiv \gamma(\hat{s};\xi) + \mathcal{U}(\hat{\Delta}; Z), \tag{10}$$

$$\gamma(s;\xi) \equiv \frac{s^2+1+v(\xi)}{2s}, \quad \frac{\mathcal{U}(\Delta;z)}{K} \equiv \int_z^{z+\Delta} d\zeta\, \widetilde{N}(\zeta) - \widetilde{N}(z)\Delta,$$

Hamiltonian $H, \gamma - 1, \mathcal{U}$ act as total, kinetic, potential energy in mc^2 units. We solve (7-8) numerically on a laptop via a *Mathematica* code, or by quadrature for $\xi \geq l$. With a slowly modulated monochromatic (SMM) wave it is $v(l) \ll 1$, whence $v(\xi) \simeq 0$, $\mathbf{u}^\perp(\xi) \simeq \mathbf{0}$, $\hat{\mathbf{x}}_e^\perp(\xi,\mathbf{X}) \simeq$const for $\xi \geq l$. The PW emerges as a collective effect going to the Eulerian description, see e.g. Fig. 4.
$\hat{z}_i, \hat{s}_i$ of a test e^- injected in the PW at $\xi_0 > l$ with $\hat{\mathbf{u}}_i^\perp = \mathbf{0}$ satisfy

$$\hat{z}_i' = \frac{1-\hat{s}_i^2}{2\hat{s}_i^2}, \quad \hat{s}_i'(\xi) = K\Big\{\widetilde{N}[\hat{z}_i(\xi)] - \widetilde{N}\Big[\hat{Z}_e(\xi, \hat{z}_i(\xi))\Big]\Big\}. \tag{11}$$

Along the plateau (11b) is $\hat{s}_i' = M\Delta$, $M \equiv K\bar{n}$. Hence

$$\hat{s}_i(\xi) = \delta s + s(\xi), \quad \hat{z}_i(\xi) = z_{i0} + \int_{\xi_0}^{\xi} \frac{dy}{2}\left[\frac{1}{\hat{s}_i^2(y)} - 1\right], \tag{12}$$

if $z_{i0} \geq z_q \equiv z_s + \Delta_M(\bar{n})$; here $(\hat{z}_i, \hat{s}_i)_{\xi=\xi_0} = (z_{i0}, s_{i0})$, $s = \hat{s}$ for $\widetilde{n_0}(z) = \bar{n}$, $\delta s \equiv s_{i0} - s(\xi_0)$. If it fulfills the **trapping condition**

$$s_i^m \equiv \bar{s}_m + \delta s < 0, \tag{13}$$

then $\hat{s}_i(\xi_f) = 0$, $\hat{s}_i'(\xi_f) = s'(\xi_f) < 0$, $\hat{t}(\xi_f) = \infty$ for some $\xi_f > \xi_0$; the e^- is **trapped in a trough of the PW and accelerated**:

$$\gamma_i \simeq \frac{F}{\lambda} z_i \xrightarrow{z_i\to\infty} \infty, \tag{14}$$

$F \equiv M\lambda|\Delta(\xi_f)|$. In this model the PW phase velocity is c, trapped e^- *cannot dephase*. (14) is reliable where pulse depletion is negligible. Fixed $z_i, \bar{n}$, e^- comove with $E^z = E_M^z$, and γ_i is maximized

$$\gamma_i(z_i, \bar{n}) \simeq \sqrt{8\pi^2\left[\bar{h}(\bar{n}) - 1\right]\bar{n}/n_{cr}}\ \frac{z_i}{\lambda}, \tag{15}$$

if ξ_0, z_0, s_0 lead to the **maximization condition**

$$\delta s = -1. \tag{16}$$

Now we can illustrate our 4-steps optimization procedure:

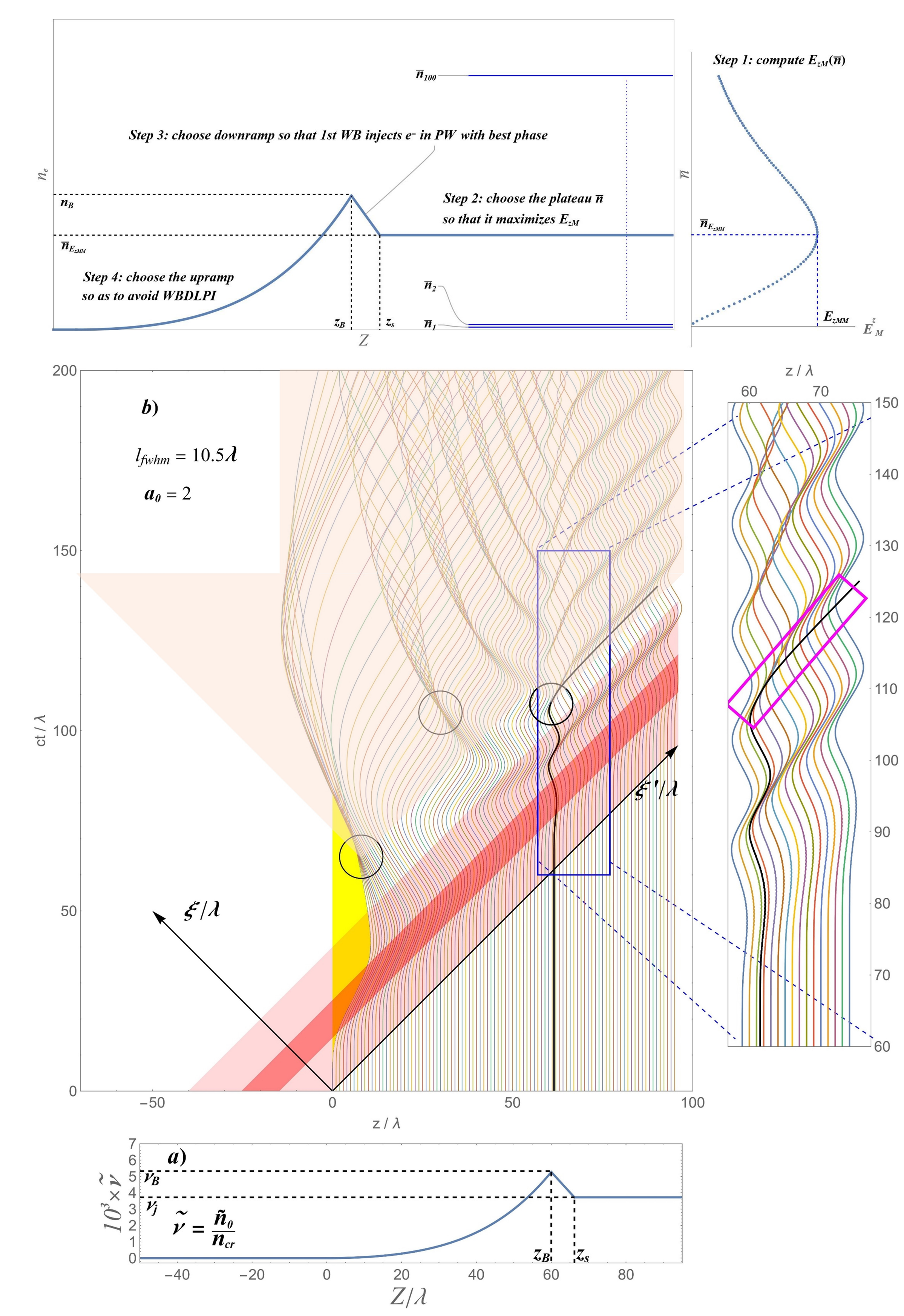


Figure 4: a) An optimal $\widetilde{n_0}(Z)$ for the pulse of Fig. 3.a: $\bar{n} = \bar{n}^j = n_{cr}/269$.
b) Corresponding solutions of (7-8) for $Z = 0, \lambda, ..., 95\lambda$: the curves are the WLs of these Z e^- *outside* the light-orange shaded causal wedges having apexes in the earliest WBs (here encircled); inside the wedges the WLs should solve (7a-9), hence may differ from the drawn curves. We have solved eqs. (7a-9) only for the Z_b e^-s, and painted their WL in black. In c) we zoom the blue box of a). Here: $\xi' \equiv ct+z$; yellow region: only ions are present; pink: support of $\boldsymbol{\epsilon}^\perp(ct-z)$ (considering $\boldsymbol{\epsilon}^\perp(\xi) = 0$ outside $0 < \xi < 40\lambda$); red: intensity FWHM region.

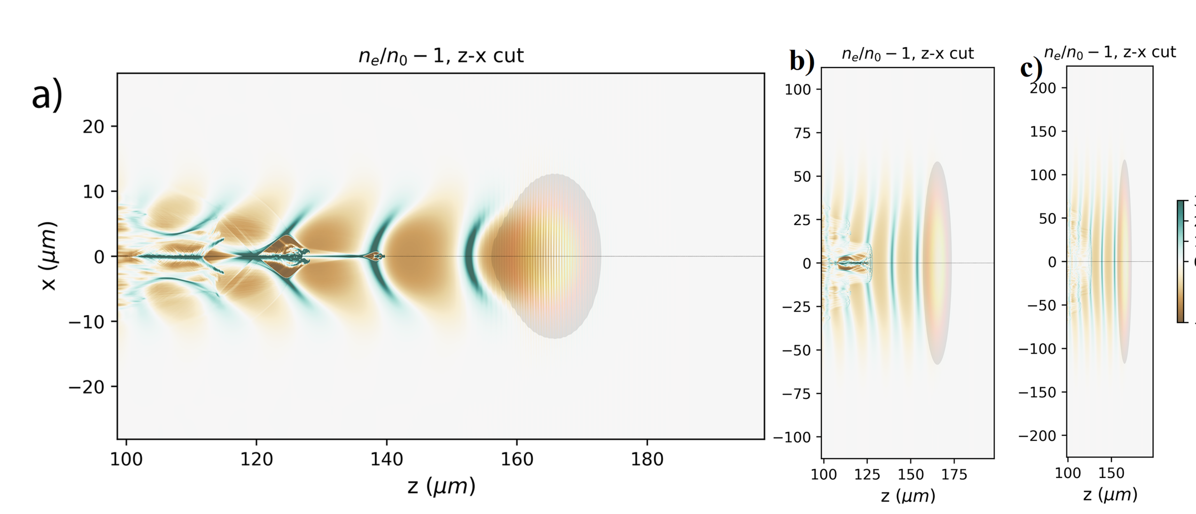


Figure 5: Quasi 3D FBPIC simulations: snapshot at $ct = 200\mu$m of charge densitiesright after the density downramp with pulse waist: a) $w_0 = 12.5\mu$m; b) $w_0 = 50\mu$m; c) $w_0 = 100\mu$m. A quasi-1D structure is evident in c).

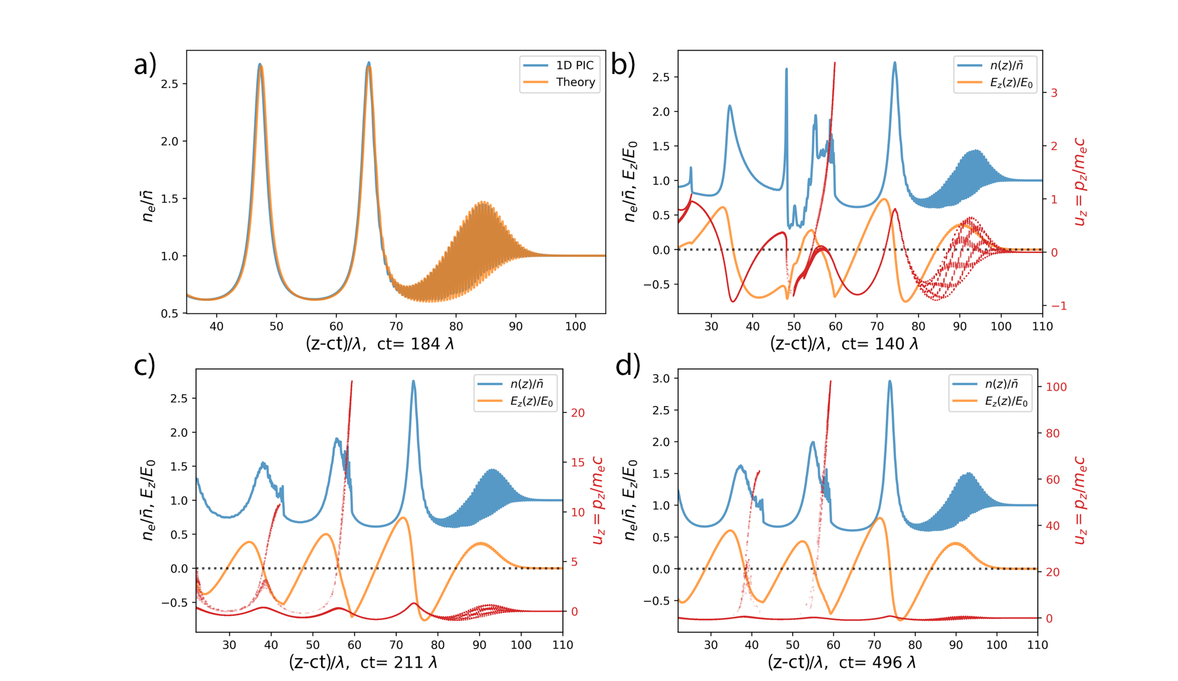


Figure 6: 1D-equivalent FB-PIC simulations run with the pulse of Fig. 3a and: a) $\widetilde{n_0}(z) = \bar{n} = n_{cr}/269$ at $ct = 184\lambda$; b)-d) the initial density $\widetilde{n_0}(z) =$ of Fig. 4, taken at 3 different times. In b)-d): the blue, orange lines are resp. the normalized plasma density, accelerating field; we plot red the longitudinal phase-space distribution of simulated particles: the self-injected e^- in the 2nd, 3rd PW bucket yield 2 steep discontinuous red lines and eroded 2nd, 3rd blue peaks.

References

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[2] G. Fiore, P. Tomassini, Physica D 2026 (in press).
[3] G. Fiore, S. De Nicola, T. Akhter, R. Fedele, D. Jovanovic, Physica D 454 (2023), 133878.